NOTE

EDGE-COLORING CLIQUES WITH THREE COLORS ON ALL 4-CLIQUEs

DHRUV MUBAYI

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A coloring of the edges of $K_n$ is constructed such that every copy of $K_4$ has at least three colors on its edges. As $n \to \infty$, the number of colors used is $O(\sqrt{\log n})$. This improves upon the previous probabilistic bound of $O(\sqrt{n})$ due to Erdős and Gyárfás.

1. The Problem

The classical Ramsey problem asks for the minimum $n$ such that every $k$-coloring of the edges of $K_n$ yields a monochromatic $K_k$. For each $n$ below this threshold, there is a $k$-coloring such that every $p$-clique receives at least 2 colors. Since the thresholds are unknown, we may study the problem by fixing $n$ and asking for the minimum $k$ such that $E(K_n)$ can be $k$-colored with each $p$-clique receiving at least 2 colors. This generalizes naturally as follows.

Definition. For integers $n, p, q$, a $(p, q)$-coloring of $K_n$ is a coloring of the edges of $K_n$ in which the edges of every $p$-clique together receive at least $q$ colors. Let $f(n, p, q)$ denote the minimum number of colors in a $(p, q)$-coloring of $K_n$.

The function $f(n, p, q)$ was first studied by Elekes, Erdős and Füredi (as described in Section 9 of [1]). Erdős and Gyárfás [2] later improved the results, using the Local Lemma to prove an upper bound of $O(n^{c_{p,q}})$, where $c_{p,q} = \frac{p-2}{(\frac{p}{q})^{q-1}}$. In addition they determined, for each $p$, the smallest $q$ such that $f(n, p, q)$ is linear in $n$, and the smallest $q$ such that $f(n, p, q)$ is quadratic in $n$. Many small cases remain unresolved, most notably the determination of $f(n, 4, 3)$. Indeed, the Local Lemma shows only that $f(n, 4, 3) = O(\sqrt{n})$, but it remains open even whether $f(n, 4, 3)/\log n \to \infty$.

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In this note we show that the optimal \((4,3)\)-coloring of \(K_n\) uses many fewer colors than the random \((4,3)\)-coloring. We do this by explicitly constructing a \((4,3)\) coloring of \(K_n\). Our main theorem is the following:

**Theorem.** \(f(n,4,3) < e^\sqrt{c \log n} (1+o(1))\), where \(c = 4 \log 2\).

## 2. The Coloring

In this section we describe the coloring of \(E(K_n)\).

We write \([n]\) for \(\{1,2,...,n\}\). The symmetric difference of sets \(A\) and \(B\) is \(A \triangle B = (A-B) \cup (B-A)\). For integers \(t < m\), let \(\binom{[m]}{t}\) denote the family of all \(t\)-subsets of \([m]\).

Let \(G\) be the complete graph on \(\binom{n}{m}\) vertices. Let \(V(G) = \binom{[m]}{t}\), and for each \(t\)-set \(T\) of \([m]\), rank the \(2^t - 1\) proper subsets of \(T\) according to some linear order. Color the edge \(AB\) with the two dimensional vector

\[ c(AB) = (c_0(AB), c_1(AB)) \]

where

\[ c_0(AB) = \min\{i : i \in A \triangle B\}. \]

Set

\[ S = \begin{cases} A & \text{if} \ c_0(AB) \in A \\ B & \text{if} \ c_0(AB) \in B. \end{cases} \]

Let \(c_1(AB)\) be the rank of \(A \cap B\) in the linear order associated with the proper subsets of \(S\).

In this construction, the number of colors used is at most \((2 - \sqrt{\log m} - 1)\).

**Remark.** This construction is valid even if we let the vertex set consist of all subsets of \([m]\) of size at most \(t\), but the gain in the number of vertices is asymptotically negligible.

## 3. The Proof

We now check that our coloring is a \((4,3)\) coloring of \(K_n\). First observe that there are no monochromatic triangles. Indeed, if \(ABC\) is one such triangle, and \(c_0(AB) = i \in A\), then, since \(c(AB) = c(BC)\) implies that \(c_0(AB) = c_0(BC)\), we have \(i \in C\). But now \(i \notin A \triangle C\), so \(c(A \triangle C) \neq c(AB)\).

Since monochromatic triangles are forbidden, the only types of 2-colored \(K_4\)'s that can occur are those in Figure 1.

**Type 1.** Here one color class is the path \(ABCD\), while the other is the path \(BDAC\). Suppose \(c_0(AB) = i\).
Case 1. \( i \in A \). Then \( i \in C \) and \( i \notin B, D \). Moreover,
\[
A \cap [i - 1] = B \cap [i - 1] = C \cap [i - 1] = D \cap [i - 1]
\]
because \( i \) is the smallest element in \( A \Delta B \) and \( c(AB) = c(BC) = c(CD) \). This implies that \( c_0(AC) > i = c_0(AD) \). Thus \( c(AC) \neq c(AD) \).

Case 2. \( i \in B \). Then \( i \notin A \), \( i \notin B, D \). Reversing the labels on the path \( ABCD \) now puts us back in Case 1.

Type 2. Here one color class is the 4-cycle \( ABCD \), while the other contains the edges \( AC \) and \( BD \). By symmetry we may assume that \( c_0(AB) \in A - B \); and hence also \( c_0(AB) \in C - D \). Thus \( c_0(AD) = c_0(AB) \in (A \cap C) - (B \cup D) \), which implies that
1) \( c_1(AB) \) is the rank of \( A \cap B \) in \( A \), and
2) \( c_1(AD) \) is the rank of \( A \cap D \) in \( A \).

Since the rank of a subset in a set identifies the subset, we have \( A \cap B = A \cap D \).

Interchanging the roles of \( A \) and \( C \), we obtain \( C \cap B = C \cap D \).

Because \( c(AC) = c(BD) \), we may assume that \( c_0(AC) = c_0(BD) = i \). Thus either \( i \in (A \cap B) - (C \cup D) \), or \( i \in (A \cap D) - (C \cup B) \), or \( i \in (C \cap B) - (A \cup D) \), or \( i \in (C \cap D) - (A \cup B) \). Each of these four cases contradicts either \( A \cap B = A \cap D \) or \( C \cap B = C \cap D \).

Proof of Theorem. Set \( t = \lfloor \sqrt{\log n} / \sqrt{\log 2} \rfloor \) and choose \( m \) such that \( t^m \leq n \leq t^{m+1} \). Since \( f \) is a nondecreasing function of \( n \) and \( (m/t)^t < (m/t)^t \) for \( t < m \), we have
\[
f(n, 4, 3) \leq f \left( \left( \frac{m + 1}{t} \right), 4, 3 \right) \leq (2^t - 1)m < 2^t n^{1/t} = (1 + o(1)) e^{\sqrt{\log 2 \log n + \frac{\log \log n - \log \log 2}{2}}} = e^{\sqrt{2 \log 2 \log n} (1 + o(1))}.
\]

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References


Dhruv Mubayi

School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia 30332-0160
mubayi@math.gatech.edu