

1. MATERIAL THAT IS NOT IN THE BOOK

Topic: In the beginning.

All math is built up from a base, a starting point, certain axioms and definitions. Modern mathematicians (19th century) asked what is the base of all math? There are several different valid ways of looking at it. They weren't concerned with it being historically accurate, but rather with all mathematics that we used today as being part of the theory that could be developed from that beginning.

2 ways emerge:

- All math can be based on elementary set theory - Dedekind cuts.
- Construct $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and properties thereof.

We will see what it is like to start with some simple axioms and definitions and see how we can derive familiar properties of real numbers from them.

Examples:

- quadratic formula
- arithmetic, geometric mean inequality
- more

We need

- (1) axioms for field
- (2) axioms for ordered field
- (3) uniqueness theorem
- (4) definition of reals

THE STRUCTURE OF OUR NUMBER SYSTEM

A short list of axioms for the real number system is given below. The reals can be constructed by starting with the natural numbers \mathbb{N} , which are described by axioms. Then the integers \mathbb{Z} are constructed from the natural numbers. Next, the rationals \mathbb{Q} and finally the reals \mathbb{R} . It can be shown that the reals, constructed by this process satisfy the axioms of a complete ordered field, given below. Also, any two structures satisfying those axioms are essentially equivalent.

So that we could alternatively define the reals as the unique complete ordered field.

Topic: Using Axioms to define a Field.

FIELD AXIOMS AND THEOREMS

Definition 1. *Field Axioms.* A set S with operations $+$ and \cdot and distinguished elements 0 and 1 with $0 \neq 1$ is a **field** if the following properties hold for all $x, y, z \in S$.

A0: $x + y \in S$. Closure of addition.

M0: $x \cdot y \in S$. Closure of multiplication.

A1: $(x + y) + z = x + (y + z)$. Associativity of addition.

M1: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$. Associativity of multiplication.

A2: $x + y = y + x$. Commutativity of addition.

M2: $x \cdot y = y \cdot x$. Commutativity of multiplication.

A3: $x + 0 = x$. Additive identity.

M3: $x \cdot 1 = x$. Multiplicative identity.

A4: Given x , there is a $w \in S$ such that $x + w = 0$. Additive inverse.

M4: Given $x \neq 0$, there is a $w \in S$ such that $x \cdot w = 1$. Multiplicative inverse.

DL: $x \cdot (y + z) = x \cdot y + x \cdot z$. Distributive Law.

The operations $+$ and \cdot are called **addition** and **multiplication**. The elements 0 and 1 are the **additive identity element** and the **multiplicative identity element**.

Examples.

Not fields. $\mathbb{N}, \mathbb{W}, \mathbb{Z}$

Fields. $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}_5$.

Use of the equal sign. We assume additionally that “ $=$ ” is an equivalence relation, that is, $a = a$, if $a = b$ then $b = a$, and if $a = b$ and $b = c$, then $a = c$. In addition to the axioms, we are making some assumptions. If $a = b$ we can substitute b for a and visa versa, thus $a + c = b + c$ by substitution.

Using parenthesis. The operations $+$ and \cdot are binary, meaning they apply to 2 field elements at a time. Thus $a + b \cdot c$ is unclear unless we assume parenthesis or include them. The convention is that \cdot comes first before $+$. So $a + b \cdot c$ is assumed to mean $a + (b \cdot c)$. If you want to say $(a + b) \cdot c$, the parenthesis are necessary.

FACTS THAT FOLLOW ALMOST IMMEDIATELY FROM THE AXIOMS

Fact 1. *The additive inverse is unique.*

Proof. Suppose $x + a = 0$ and $x + b = 0$. Then

$$(x + a) + b = 0 + b = b.$$

On the other hand,

$$(x + a) + b = x + (a + b) = x + (b + a) = (x + b) + a = 0 + a = a.$$

Now we have $a = b$, since they are each equal to the same expression. \square

Fact 2. *The multiplicative inverse is unique.*

Proof. You prove this. \square

Fact 3. *For all x ,*

- (1) $0 + x = x$, and
- (2) if $x \neq 0$, then $1 \cdot x = x$.

Proof. You prove this. \square

Fact 4. *For all $x \in F$, $x - x = 0$.*

Proof. You prove this. \square

Notice that we were careful in the above arguments to use only the axioms.

Definition 2. *The **negative** of x is the unique additive inverse and is written $-x$. The **reciprocal** of x is the unique multiplicative inverse and is written x^{-1} . We define **subtraction** as the addition of the additive inverse. We define **division** of x by y as the multiplication of x by the reciprocal of y . Notice, 0 has no reciprocal and we therefore cannot divide by 0 . We say xy instead of $x \cdot y$.*

For the following, assume F is a field and $x, y, z \in F$.

Proposition 1. .

- (1) $x + z = y + z$ implies $x = y$
- (2) $x \cdot 0 = 0$
- (3) $(-x)y = -(xy)$
- (4) $-x = (-1)x$
- (5) $(-x)(-y) = xy$
- (6) $xz = yz$ and $z \neq 0$ imply $x = y$
- (7) $xy = 0$ implies $x = 0$ or $y = 0$

Proof. You prove all of these, using only the axioms. \square