Set Representations of Graphs

Some terms

February 5, 2008

A set representation of a graph consists of a family of sets, and a rule which determines when 2 vertices form an edge.

The following example is historically the first:

**Definition 1** Let $G = (V, E)$ be a graph with $|V| = n$. A family $\mathcal{F} = \{A_x : x \in V\}$ of (not necessarily distinct) sets is called an intersection representation of $G$ if

$$A_x \cap A_y \neq \emptyset \iff \{x, y\} \in E$$

for every pair $x, y$ of distinct vertices of $G$.

We are interested in finding the minimum size of a universe in which a graph can be represented. Thus we have the following definition:

**Definition 2**

$$\theta_1(G) = \min_{\mathcal{F}}(|\cup \mathcal{F}|)$$

where the minimum is taken over all intersection representations $\mathcal{F}$.

Notice $\theta_1(G) \leq |E|$ since for $A_x = \{e \in E : x \in e\}$,

$$\mathcal{F} = \{A_x : x \in V\}$$

gives an intersection representation of $G$ such that $|\cup \mathcal{F}| \leq |E|$.

One variation on intersection representations is the following:

**Definition 3** Let $G = (V, E)$ be a graph. A family $\mathcal{F} = \{A_x : x \in V\}$ of (not necessarily distinct) sets is called a $p$-intersection representation of $G$ if

$$|A_x \cap A_y| \geq p \iff \{x, y\} \in E$$

for every pair $x, y$ of distinct vertices of $G$. Then we define

$$\theta_p(G) = \min_{\mathcal{F}}(|\cup \mathcal{F}|)$$

where $\mathcal{F}$ is taken over all $p$-intersection representations of $G$. 

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Clearly, for any graph $G$ and integer $p \geq 2$, a $p$-intersection representation of $G$ exists since we may use an intersection representation and add 1 element to the universe and include it in each set. This gives $\theta_p(G) \leq \theta_1(G) + p - 1$. But we know in some cases, one can do much better.

**Characterization**

There is a relationship between vertex coverings of a graph and intersection representations.

**Definition 4** A $p$-edge clique cover of a graph $G = (V, E)$ to be a family, $\mathcal{C}$, of subsets of $V$ such that $e \in E \iff |\{C \in \mathcal{C} : e \subseteq C\}| \geq p$.

$$\mathcal{F} \subseteq \mathcal{C}, \quad \theta_p = \min_{\mathcal{C}} |\mathcal{C}|.$$  

Even more general than the $p$-intersection representation, we have,

**Definition 5** Given a graph $G = (V, E)$, a family $\mathcal{F} = \{S_u : u \in V\}$ and a set $L$ such that $|S_u \cap S_v| \in L \iff \{u, v\} \in E$.

We say $\text{gdim}(G) = f(n)$ when $f(n) = \min_{\mathcal{F}} |\bigcup \mathcal{F}|$ over all possible general representations $\mathcal{F}, L$. 

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**Defn.**

$$\theta_p(n) = \max \left\{ \Theta_p(G) \mid |V(G)| = n \right\}$$
Representations of Graphs By Sets

A representation of a graph consists of a family of sets, and a rule which determines when 2 vertices form an edge.

The following example is historically the first:

Ex 1: Let \( G = (V, E) \) be a graph with \(|V| = n\).
(In fact, throughout this talk, assume \( G \) always has \( n \) vertices.)

Definition 1 A family \( \mathcal{F} = \{A_x : x \in V\} \) of (not necessarily distinct) sets is called an intersection representation of \( G \) if

\[
A_x \cap A_y \neq \emptyset \iff \{x, y\} \in E
\]

for every pair \( x, y \) of distinct vertices of \( G \).

We are interested in finding the minimum size of a universe in which a graph can be represented. Thus we have the following definition:

\[
\theta_1(G) = \min_{\mathcal{F}}(|\cup \mathcal{F}|)
\]

where the minimum is taken over all intersection representations \( \mathcal{F} \).

Notice \( \theta_1(G) \leq |E| \) since for \( A_x = \{e \in E : x \in e\} \),

\[
\mathcal{F} = \{A_x : x \in V\}
\]

gives an intersection representation of \( G \) such that \(|\cup \mathcal{F}| \leq |E|\).

Characterization

There is a relationship between coverings of the edges of a graph by cliques and intersection representations. We define an edge clique cover to be a family, \( \mathcal{C} \), of cliques such that \( E = \cup \mathcal{C} \).
So we see that
\[ \theta_1(G) = \min(|C|) \]  
where the minimum is taken over all clique covers of \( G \).

We know much more than this. Erdős, Goodman and Pósa ('64) showed

**Theorem 1** For all \( G \),
\[ \theta_1(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor \]
and this is the best possible in the sense that there exists a graph \( G \) such that
\[ \left\lfloor \frac{n^2}{4} \right\rfloor = \theta_1(G). \]

One can see that for the graph \( G = K_{\left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor} \)
\[ \theta_1(G) = \left\lfloor \frac{n^2}{4} \right\rfloor \]
since \( K_{\frac{n}{2}, \frac{n}{2}} \) has no triangles so in order to cover the edges by cliques, we must always use single edges.

Another natural question to ask at this point is, given a graph \( G \), is there a fast algorithm to find \( \theta_1(G) \)? The answer is no due to the following

**Theorem 2** (Kou, Stockmeyer and Wong '78) For a given graph \( G \) and integer \( t \) it is NP-complete to decide whether \( \theta_1(G) \leq t \).

I didn't mention last time
Proof of Thm 1:

Suppose $G$ has $n$ verts.

We wish to show $t \leq \left\lfloor \frac{n^2}{4} \right\rfloor$

Let $G$ have edge $uv = e$

Set $H = G - \{u,v\}$ so that $|V(H)| = n - 2$

By induction $\Delta(H) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor$

and $\sum C_i \ldots, C_e$ clique cover of $H$.

Each vertex of $H$ is either adjacent to both $u$ and $v$, just $u$, just $v$, or neither.
If both make a \( \Delta x \) on an edge \( u \) or \( v \)

Finally in the worst case, add one edge

\[
\begin{align*}
\ell + n - 1 & \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + n - 1 \\
\end{align*}
\]

\[\text{Case 1:} \quad n \text{ even.} \]

\[n = 2k \Rightarrow \left\lfloor \frac{(n-2)^2}{4} \right\rfloor = k^2 - n + 1 \]

\[\ell + n - 1 \leq k^2 - n + 1 + n - 1 = k^2 \]

\[\left( \frac{n}{2} \right)^2 = \frac{n^2}{4} = \left\lfloor \frac{n^2}{4} \right\rfloor \]

\[\text{Case 2:} \quad n \text{ odd} \]

\[n = 2k + 1 \]

\[\left( \frac{n-2}{2} \right)^2 = \left( \frac{4k^2 + 4k + 1 - 4n + 4}{4} \right) \]

\[= k^2 + k - n + 1 + \frac{1}{4} \]

\[\left[ \frac{n-2}{2} \right]^2 = k^2 + k - n + 1 \]

\[\ell + n - 1 \leq k^2 + k - n + 1 + n - 1 = k^2 + k \]
and \( \frac{n^2}{24} = \frac{4k^2 + 4k + 1}{4} \) so \( \left\lfloor \frac{n^2}{4} \right\rfloor = k^2 + k \)

\[ 0 \leq n - 1 \leq k^2 + k = \left\lfloor \frac{n^2}{4} \right\rfloor \]

Also if \( G = \left\lfloor \frac{n_1}{2} \right\rfloor, \left\lfloor \frac{n_2}{2} \right\rfloor \)

there are no \( \Delta \) so we need each edge in a clique (over

So \( \Theta(G) = \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n}{2} \right\rfloor \)

Case 1: \( n \) even \( \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n^2}{2} \right\rfloor = \frac{n}{2} \) and

\( \left\lfloor \frac{n^2}{4} \right\rfloor = \frac{n^2}{4} \).

Case 2: \( n \) odd \( \left\lfloor \frac{n}{2} \right\rfloor = \frac{n-1}{2} \) and \( \left\lfloor \frac{n}{2} \right\rfloor = \frac{n+1}{2} \)

product = \( \frac{n^2 - 1}{4} = \left\lfloor \frac{n^2}{4} \right\rfloor \)
Ex 2:

SLIDES:

One variation on intersection representations is the following:

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$$|A_x \cap A_y| \geq p \iff \{x, y\} \in E$$

for every pair $x, y$ of distinct vertices of $G$. Then we define

$$\theta_p(G) = \min_{\mathcal{F}}(|\cup \mathcal{F}|)$$

where $\mathcal{F}$ is taken over all $p$-intersection representations of $G$.

Clearly, for any graph $G$ and integer $p \geq 2$, a $p$-intersection representation of $G$ exists since we may use an intersection representation and add 1 element to the universe and include it in each set. This gives $\theta_p(G) \leq \theta_1(G) + p - 1$. But we know in some cases, one can do much better.
But here, not as much is known. Chung and West showed recently

**Theorem 3** For $G = K_{\frac{n}{2}, \frac{n}{2}}$, and $p \geq 2$,

$$\theta_p(G) \geq \frac{n^2 + 2(2p - 1)n}{4p}.$$  

**Theorem 4** For $n$ odd, $n \not\equiv 0 \pmod{3}$,

$$\theta_2(K_{\frac{n}{2}, \frac{n}{2}}) = \frac{n^2 + 6n}{8}.$$  

But it is unknown for instance if there exists a graph $G$ with $\theta_2(G) > \theta_2(K_{\frac{n}{2}, \frac{n}{2}})$. This appears to be a hard question.

I proved there is such a graph.