On Two **Short Proofs** About List Coloring Graphs

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LIST COLORINGS AND CHOICE NUMBER

DEF: A $k-list\ assignment$, L, is an assignment of sets (called $lists$) to the vertices so that

$$|L(v)| \geq k,$$

for all vertices v, an $L-list\ coloring$ is a coloring such that the color assigned to v is in $L(v)$ for all vertices v. If G is such that a coloring exists for all possible k-list assignments, we say that G is $k\ choosable$. The smallest k for which G is k choosable is the $choice\ number$ of G, denoted $ch(G)$.

NOTICE: A k coloring is an L-list coloring where all lists are $\{1, 2, \ldots, k\}$.

FACT: For all graphs on n vertices,

$$\chi(G) \leq ch(G) \leq \chi(G) \ln(n)$$

.

Warm up

We show $ch(G) \leq \chi(G) \ln |V(G)|$

A probabilistic argument.

Color G with $s = \chi(G)$ colors,

Color classes: $C_1, C_2, \ldots, C_s$

Suppose $k = \chi(G) \ln n$, where $n = |V(G)|$.

Assume G has a k-list assignment L.

The sample space is the set of all partitions of $\cup_{u \in V(G)} L(v)$ into at most s parts. A typical partition is $P : P_1, P_2, \ldots, P_s$ and

For $c \in \cup_{u \in V(G)} L(v)$, Prob $(c \in P_i) = \frac{1}{s}$

Want $\forall i \in [s], \forall v \in C_i, L(v) \cap P_i \neq \emptyset$.

Prob $(\exists i \in [s], \exists v \in C_i, L(v) \cap P_i = \emptyset)$

$\leq \sum_{v \in V(G)} \left(\frac{s-1}{s}\right)^k = n((1 - \frac{1}{s})^2)^{\frac{k}{s}} < ne^{-\frac{k}{s}}$.

We see that $s \ln n = k$

$\ln n = \frac{k}{s}$

$n = e^{\frac{k}{s}}$

$ne^{-\frac{k}{s}} = 1$

And so Prob $(\exists i \in [s], \exists v \in C_i, L(v) \cap P_i = \emptyset)$

$< ne^{-\frac{k}{s}} = 1$

Therefore, the probability

$\forall i \in [s], \forall v \in C_i, L(v) \cap P_i \neq \emptyset > 0$.

So there exists such a partition.

Grötzsch (B) $\Rightarrow$ (A)

(A) Grotzsch: Every planar graph G of girth at least 4 is 3-colorable.

Moreover, if G has an outer cycle of length 4 or 5 then any 3-coloring of the outer cycle can be extended to a 3-coloring of G.

(B) Grotzsch's girth 5 version: Every planar graph G of girth at least 5 is 3-colorable.

Moreover, if G has an outer cycle of length 5 then any 3-coloring of the outer cycle can be extended to a 3-coloring of G.

Use (B) to prove (A).

Grötzsch (B) $\Rightarrow$ (A)

Proof by induction on $|V(G)|$.

If G has no 4 cycles then by (B) we are done.

Assume G has 4 cycles.

If G has a vertex v of degree at most 2, color $G-v$ by induction, and color v afterward with a color not used on either of its neighbors.

So assume all degrees are at least 3.

If G is disconnected, use induction on each component.

If G has a cut vertex v, such that $G-v$ has components, $C_1, C_2, \ldots, C_k$, color each of $C_i + v$ using induction then permute the colors if necessary so that v is colored the same in each.

Now we assume that G is 2-connected. So that every facial walk is a cycle.

Suppose the length of C is greater than 5. Then it is not precolored. We find a facial cycle of length at most 5 and make that the outer cycle and precolor it.

We know such a facial cycle exists by Euler's formula and the fact that we know for all v, the degree of v is at least 3.

Let f be the number of faces, n the number of vertices, and e the number of edges.

$$\sum_{i=1}^{n} deg(v_i) = 2e, \quad 3n \leq 2e$$

$$n - e + f = 2, \quad n = 2 - f + e$$

$$3(2 - f + e) \leq 2e, \quad e \leq 3f - 6$$

But if $\forall i$, $deg(f_i) \geq 6$ then

$$2e = \sum_{i=1}^{f} deg(f_i) \geq 6f$$

and $e \geq 3f$. This is a contradiction.

So we assume we have outer cycle C of length 4 or 5 and it is precolored.

If G has a separating 4 or 5 cycle C'. We color $ext(C') \cup C'$ first using induction then color $int(C') \cup C'$ by induction.

2 pictures:

If G has a vertex joined to two vertices of C we precolor that vertex in $int(C)$ and use induction on the 2 parts.

2 pictures:

If G has a facial 4-cycle distinct from C, identify 2 opposite vertices and use induction to color the resulting graph, then reverse the process and the same color is valid for both of the original vertices.

If identifying the 2 opposite vertices cause a chord in C that would give a 3 cycle and we could not use induction, then the other 2 opposite vertices of the 4-cycle will not.

Notice that we cannot simply add 5th vertices to all the 4 cycles of G and apply (B).

Otherwise, we can assume that C is the only 4-cycle. We CAN insert a new vertex on C and precolor it. Now we apply (B)

Picture

Thomassen's Long proof

Let G be a planar graph of girth at least 5. Let A be a set of vertices in G such that each vertex of A is on the outer cycle. Assume that either

(i) $G(A)$ has no edge or

(ii) $G(A)$ has precisely one edge xy and G has no 2-path from x to a vertex of A.

Assume that L is a color assignment such that $|L(v)| \geq 2$ for each vertex in G and $|L(v)| \geq 3$ for each vertex in $V(G) \backslash A$. Let u, w be any adjacent vertices in G both on the outer face boundary and let $c(u), c(w)$ be distinct colors in $L(u)$ and $L(w)$ respectively. Then c can be extended to a list coloring of G.

Thomassen's Long proof $\Rightarrow$ (B)

Assume G has girth at least 5 with outer cycle C and $|C| > 5$ then C is not precolored so we give every vertex the same list of size 3 and use Thomassen.

Assume $C : v_1, v_2, v_3, v_4, v_5, v_1$ is precolored.

Let v_1 and v_2 play the role of u and w.

$A = \{v_3, v_5\}$

$L(v_3) = \{c(v_3), c(v_2)\}$

$L(v_5) = \{c(v_5), c(v_1)\}$

$L(v_4) = \{c(v_3), c(v_4), c(v_5)\}$

Thomassen's Short proof

Let G be a plane graph of girth at least 5. Let c be a 3-coloring of a path or cycle P : $v_1, v_2, \ldots, v_q$, $1 \leq q \leq 6$ such that all vertices of P are on the outer face boundary.

For all $v \in V(G)$, let $L(v)$ be its list of colors. If $v \in P$ then $L(v) = \{c(v)\}$. Otherwise $|L(v)| \geq 2$. If v is not on the outer face boundary then $|L(v)| = 3$.

There are no edges joining vertices whose lists have at most 2 colors, except the edges of P.

Then c can be extended to an L-coloring of G.

Thomassen's Short proof $\Rightarrow$ (B)

Easier.

Proof of ”Thomassen’s Short proof”

By induction on $n = |V(G)|$

Assume G is connected.

Assume no cut vertices in P.

Assume no cut vertex at all.

Assume no edge of P is a chord of C.

Let $C : v_1, v_2, \ldots, v_q, v_{q+1}, \ldots, v_k, v_1$.

(1)

We can assume $P \neq C$ and $q + 3 \leq k$.

Since if $P{+}C$ we could use induction by taking some v_i remove $c(v_i)$ from all 3 lists of the neighbors. Remember girth $= 5$ so no vertices with new lists of size 2 will be adjacent to each other and if one is adjacent to a vertex in P we can break into two parts and use induction.

Similarly if $k < q + 3$.

Some pictures:

(2)

C has no chord.

(3)

If $q \leq 6$ then there is no $v_i u v_q$ path with $u \in int(C)$. Except if $q = 6$ then there could be a $v_4 u v_7$ or $v_3 u v_i$.

(4)

G has no path of the form $v_j u w v_i$ where $u, w \in int(C)$ and $L(v_i)$ has exactly 2 colors.

G has no path $v_j u w v_i$ where $u, w \in int(C)$ and $L(v_i)$ has 3 colors and $j \in \{1, q\}$.

(5)

If C' is a cycle in G with at most 6 vertices distinct from C then the interior of C' is empty.

Finish off

A theorem of E, and Hull

DEF: A $defective\ coloring$ with defect d is a coloring of the vertices such that for each color class C, the maximum degree of the induced graph on C is d. If there exists a k-coloring of a graph is defective with defect d, we say the graph is $(k, d) - colorable$.

NOTICE: A defective coloring with defect 0 is a proper coloring. And for all d, and all k, if a graph is k colorable then it is d-defective k colorable.

Theorem: All planar graphs are 3-list colorable with defect 2.

In the spirit of Thomassen

THM: Given a planar graph G with outer circuit $C = (v_1, v_2, \ldots, v_k)$, and list assignment L such that for $v \in V(C)$, $|L(v)| \geq 2$ and otherwise $|L(v)| \geq 3$. For any pre-coloring c of the vertices v_1 and v_2, this coloring can be extended to a 1-defective L-coloring of G in such a way that

- If $c(v_1) = c(v_2)$ then $def(v_1) = def(v_2) = 1$.
- If $c(v_1) \neq c(v_2)$ then $def(v_1) \leq 1$ and $def(v_2) = 0$.

QUESTION:

Are all planar graphs (4,1)-choosable?