On Two **Short Proofs** About List Coloring Graphs

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CHROMATIC NUMBER

DEF: A $k-coloring$ of a graph G is a function $c : V(G) \rightarrow \{1, 2, \ldots k\}$. A $proper\ k-coloring$ of a graph G is a coloring of G with k colors so that no 2 distinct adjacent vertices are the same color.

The $chromatic\,number$ of G, $\chi(G)$, is the smallest k such that a proper k-coloring of G exists.

LIST COLORINGS AND CHOICE NUMBER

DEF: A $k - list\ assignment$, L, is an assignment of sets (called $lists$) to the vertices so that

$$|L(v)| \geq k,$$

for all vertices v, an $L - list\ coloring$ is a coloring such that the color assigned to v is in $L(v)$ for all vertices v. If G is such that a coloring exists for all possible k-list assignments, we say that G is k $choosable$. The smallest k for which G is k choosable is the $choice\ number$ of G, denoted $ch(G)$.

NOTICE: A k coloring is an L-list coloring where all lists are $\{1, 2, \ldots, k\}$.

FACT: For all graphs on n vertices,

$$\chi(G) \leq ch(G) \leq \chi(G) \ln(n)$$

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EXAMPLES OF LIST COLORING

Planar Graphs

DEF: A graph is $planar$ if it can be drawn in the plane with no edge crossings.

DEF: A graph is $bipartite$ if and only if its chromatic number is 2.

Chronology:

1. Chromatic Number

   (a) Heawood 1890, 5-color theorm

       For planar graphs

   (b) Grötzsch 1959, 3-color theorem

       For planar graphs of girth 4

   (c) Grünbaum 1962, 3-color theorem

       For planar graphs with at most 3 3-cycles

   (d) Appel, & Haken 1976, 4-color theorem

       For planar graphs

   (e) N. Robertson, D. P. Sanders, P. D. Seymour and R. Thomas, 1996, 4-color theorem

       For planar graphs

2. List Coloring and Choice Number

(a) Alon & Tarsi 1992, 3-choosable
For planar and bipartite

(b) Thomassen 1994, 5-choosable
For planar graphs
Implies (1a)

(c) Thomassen 1995, 3-choosable,
For planar graphs of girth 5
Implies (1b)

(d) Thomassen 2003, 3-choosable,
For planar graphs of girth 5
A short proof

Thomassen 5-color

1994 Thomassen

THM: If G is planar then the choice number is at most 5.

He proved something stronger:

THM: Let G be planar with outercircuit $C = (v_1, v_2, \ldots, v_k)$, and let L be a list assignment such that for $v \in V(C)$, $|L(v)| \geq 3$ otherwise $|L(v)| \geq 5$. For any precoloring, c, of the vertices v_1 and v_2, c can be extended to an L-coloring of G.

Proof

Thomassen's implies Grotzsch

(A) Grotzsch: Every planar graph G of girth at least 4 is 3-colorable.

Moreover, if G has an outer cycle of length 4 or 5 then any 3-coloring of the outer cycle can be extended to a 3-coloring of G.

(B) Grotzsch's girth 5 version: Every planar graph G of girth at least 5 is 3-colorable.

Moreover, if G has an outer cycle of length 5 then any 3-coloring of the outer cycle can be extended to a 3-coloring of G.

Use (B) to prove (A).

Thomassen's Long proof

Let G be a planar graph of girth at least 5. Let A be a set of vertices in G such that each vertex of A is on the outer cycle. Assume that either

(i) $G(A)$ has no edge or

(ii) $G(A)$ has precisely one edge xy and G has no 2-path from x to a vertex of A.

Assume that L is a color assignment such that $|L(v)| \geq 2$ for each vertex in G and $|L(v)| \geq 3$ for each vertex in $V(G) \backslash A$. Let u, w be any adjacent vertices in G both on the outer face boundary and let $c(u), c(w)$ be distinct colors in $L(u)$ and $L(w)$ respectively. Then c can be extended to a list coloring of G.

Thomassen's Short proof

Let G be a plane graph of girth at least 5. Let c be a 3-coloring of a path or cycle P : $v_1, v_2, \ldots, v_q$, $1 \leq q \leq 6$ such that all vertices of P are on the outer face boundary.

For all $v \in V(G)$, let $L(v)$ be its list of colors. If $v \in P$ then $L(v) = \{c(v)\}$. Otherwise $|L(v)| \geq 2$. If v is not on the outer face boundary then $|L(v)| = 3$.

There are no edges joining vertices whose lists have at most 2 colors, except the edges of P.

Then c can be extended to an L-coloring of G.

Grotzsch's girth 5 version

Notice that both of these imply Grotzsch's girth 5 version.

General Graphs

1979 Erdös, Rubin and Taylor

- Characterized all 2-choosable graphs

- Showed that for every n, there exists a c such that $ch(K_{n,n}) > c \ln n$.

- If G is connected, not K_n, not and odd cycle then $ch(G)$ is at most its maximum degree.

Line Graphs

DEF: The *line graph* of a graph G is the graph whose vertices are the edges of G and two are adjacent if and only if their corresponding edges in G share and endpoint.

In 1985, the following mathematicians, Vizing, Albertson, Collins, Tucker, Gupta, Bollobás and Harris, made the following conjecture.

CONJ: For all G, the choice number of the line graph of G is equal to the chromatic number of the line graph of G.

In 1995, Galvin proved the conjecture for bipartite graphs. This fact was then used to solve a well known conjecture of Dinitz (1979). He used algebraic techniques of Alon & Tarsi.

Tomaž Slivnik 1996, **A short proof** of Galvin's theorem on the list-chormatic index of a bipartite multigraph.

DINITZ' (THEOREM)

THM: Given an $n \times n$ array, and for each i, j, a set, $A_{i,j}$ of n numbers, there exists an assignment of a number to each position where for position i, j the number comes from set $A_{i,j}$ and there are no repeats in any row or in any column.