(1) We use induction to prove the generalized distributive law, that is

\[ \sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i \]

**Proof.** Base Case: \( n = 2 \) this is the ordinary distributive law. We know that \( ca_1 + ca_2 = c(a_1 + a_2) \).

**Induction Assumption:** Assume that \( k \geq 2 \) and \( \sum_{i=1}^{k} ca_i = c \sum_{i=1}^{k} a_i \) is true.

**Now we prove the statement for** \( k + 1 \)

\[
\sum_{i=1}^{k+1} ca_i = \sum_{i=1}^{k} ca_i + ca_{k+1} \\
= c \sum_{i=1}^{k} a_i + ca_{k+1}, \text{by the induction hypothesis} \\
= c \left( \sum_{i=1}^{k} a_i + a_{k+1} \right), \text{by the ordinary distributive law} \\
= c \sum_{i=1}^{k+1} a_i. 
\]

\[ \square \]

Use the generalized distributive law and the formula \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \) to find formulas for:

(a) \( \sum_{k=1}^{n} (2k + 1) \)
(b) \( \sum_{k=1}^{n} (4k - 1) \)
(c) \( \sum_{k=0}^{n} (4k + 1) \)
(d) \(-1 + 2 - 3 + 4 - \cdots - (2n - 1) + 2n\)

(2) Prove by induction:

\[ \sum_{i=1}^{n} (-1)^i i^2 = (-1)^n \frac{n(n+1)}{2}. \]

(3) Prove that \( 2^n \leq n! \) for all \( n \in \mathbb{N}, n \geq 4. \)
(4) Prove by induction: If \( q \in \mathbb{R}, q \neq 1, \) and \( n \) is a nonnegative integer, then
\[
\sum_{i=0}^{n-1} q^i = \frac{q^n - 1}{q - 1}
\]

(5)
\[
\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}
\]

(6) Show that \( 0 \leq a_i \leq b_i, \forall i \in [n] \) implies
\[
\prod_{i=1}^{n} a_i \leq \prod_{i=1}^{n} b_i
\]

(7) Find the flaw in the following “proof”.
We will prove that for all \( a, n \in \mathbb{N} \cup \{0\}, a^n = 1 \)

Proof. The base case is for \( n = 0 \). We know that \( a^0 = 1 \).
Assume true for \( k \). That is, \( a^k = 1 \).
Notice that
\[
a^{n+1} = a^{2n-(n-1)} = \frac{a^{2n}}{a^{n-1}} = \frac{a^n a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1.
\]
\(\square\)

(8) Prove that if \( x \) and \( y \) are distinct real numbers, then \( (x + 1)^2 = (y + 1)^2 \) if and only if \( x + y = -2 \). How does the conclusion change if we allow \( x = y \)?

Proof: We start with \( (x + 1)^2 = (y + 1)^2 \). Expanding both sides, we get \( x^2 + 2x + 1 = y^2 + 2y + 1 \). Cancelling the 1’s and rearranging terms, we get \( x^2 - y^2 = 2y - 2x \). We factor each side to get \( (x + y)(x - y) = 2(y - x) = -2(x - y) \). As we are given \( x \neq y \), we can divide both sides by \( x - y \) to get \( x + y = -2 \).

As equations preserve equality and these steps are reversible, \( x + y = -2 \) implies \( (x + 1)^2 = (y + 1)^2 \), so the dependence holds both ways.

If \( x \) and \( y \) are allowed to be the same we would say: \( (x + 1)^2 = (y + 1)^2 \) if and only if either \( x = y \) or \( x + y = -2 \).

(9) Given a real number \( x \), let \( A \) be the statement \( 1/2 < x < 5/2 \), let \( B \) be the statement \( x \in \mathbb{Z} \), let \( C \) be the statement \( x^2 = 1 \), and let \( D \) be the statement \( x = 2 \). Which statements below are true for all \( x \in \mathbb{R} \).
(a) \( A \Rightarrow C \). False. For a counterexample, let \( x = 2 \).
(b) \( B \Rightarrow C \). False. For a counterexample, let \( x = 2 \).
(c) \( (A \land B) \Rightarrow C \). False. For a counterexample, let \( x = 2 \).
(d) \((A \land B) \Rightarrow (C \lor D)\). True. If \(A\) and \(B\) are true, then \(x = 1\) or \(x = 2\). If \(x = 1\) then \(C\) is true. If \(x = 2\) then \(D\) is true. So in either case, either \(C\) or \(D\) is true.

(e) \(C \Rightarrow (A \land B)\). False. For a counterexample let \(x = -1\).

(f) \(D \Rightarrow [A \land B \land (\neg C)]\). True. If \(x = 2\) then \(A\) is true, \(B\) is true, and \(x^2 \neq 1\) so \(\neg C\) is true.

(g) \((A \lor C) \Rightarrow B\). True. \(A \lor C\) means that \(x = 1, 2,\) or \(-1\). So, \(B\) is true.

(10) Let \(x, y\) be integers. Determine the truth value of each statement below.

(a) \(xy\) is odd if and only if \(x\) and \(y\) are odd. TRUE. If \(x\) is odd and \(y\) is odd then \(x = 2a + 1\) for some \(a\) and \(y = 2b + 1\) for some \(b\). Then \(xy = (2a + 1)(2b + 1) = 4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1\) which is odd. Suppose it is not true that both \(x\) and \(y\) are odd. Then at least one is even. Without loss of generality, suppose \(x\) is even. Then \(x = 2a\) for some \(a\). \(xy = 2ay\) which is even.

(b) \(xy\) is even if and only if \(x\) and \(y\) are even. FALSE. For a counterexample to the “only if” part, let \(x = 3\) and \(y = 4\). Then, \(xy = 12\) which is even.

(11) Prop1 (2): \(x \cdot 0 = 0\)
Try starting with \(0 + 0 = 0\) which is true for what reason?

(12) Prop1 (3): \((-x)y = -(xy)\)
Try showing that \(xy + (-x)y = 0\). This would imply that \((-x)y\) is the unique additive inverse of \(xy\), which is the same as \(-(xy)\) by definition.

(13) Prop1 (4): \(-x = (-1)x\) Try using (3). First Rewrite (3) with different letters to avoid confusion. \((-s)t = -(st)\). Now substitute \(s = 1\) and \(t = x\).

(14) Prop1 (5): \((-x)(-y) = xy\). For this one, you might need that \(-(-z) = z\). So prove that first. Then use it and (3) to prove (5).

(15) Prop1 (6): \(xz = yz\) and \(z \neq 0\) imply \(x = y\). Multiply both sides by \(z^{-1}\).

(16) Prop1 (7): \(xy = 0\) implies \(x = 0\) or \(y = 0\).

\textbf{Proof. Case 1:} \(x = 0\)
We are done since \(x = 0\) or \(y = 0\) holds.

\textbf{Case 2:} \(\neq 0\)
In this case \(x^{-1}\) exists.
\[ xy = 0, \text{ given} \]
\[ x^{-1}(xy) = x^{-1}0, \text{ Property of "="} \]
\[ (x^{-1}x)y = x^{-1}0, \text{ Assoc of mult.} \]
\[ (x^{-1}x)y = 0, \text{ Prop1 (2)} \]
\[ 1 \cdot y = 0, \text{ Multiplicative inverse} \]
\[ y = 0, \text{ Multiplicative identity} \]
Thus we showed either \( x = 0 \), or \( y = 0 \). □