Tolerance Representations of Graphs in Trees
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Tolerance Representations of Graphs in Trees
Lecture I

• Tree Representations of Graphs

• Tolerance Representations

• Some background

• A conjecture on Tolerance Representations of Graphs in Trees and related result

Tolerance Representations of Graphs in Trees
Lecture II
• Using Special families of Host trees
  – Paths
  – $k$-ary trees
  – Asters
  – Caterpillars

• A Theorem - Representing the Complete bipartite graph in a binary tree

• Using special families of Subtrees

• Many directions to explore

Tree representations of graphs

Given a tree $T$ and a graph $G$, 
a $T$-representation of $G$

is an assignment $u \mapsto S_u$ of subtrees of $T$ to the vertices of $G$ so that

$$\{u, v\} \in E(G) \iff V(S_u) \cap V(S_v) \neq \emptyset$$

The tree $T$ is called the **host tree** in the representation. A graph which has such a representation is called a **subtree graph**. Keep in mind that each $S_u$ is a subtree of $T$, so it is a connected subgraph of $T$.

**Question 1** Do all graphs have tree representations? Ans: No.

**Tolerance Representations**

Given a host tree $T$ and a graph $G$ we say that the assignment of subtrees of $T$ to the vertices of $G$, $v \mapsto S_v$
is a $T$ tolerance representation of $G$ with tolerance $t$ if

$$\{u, v\} \in E(G) \iff |V(S_u) \cap V(S_v)| \geq t$$

**Question 2** Does every graph have a tree tolerance representation for some tolerance at least 2? Ans: Yes.

**Path representations**

The host tree is a path.

**Definition 1** An interval representation of a graph $G$ is an assignment,

$$u \mapsto I_u$$

of intervals on the real line to the vertices of the graph so that

$$u, v \in E(G) \iff I_u \cap I_v \neq \emptyset$$
A graph with an interval representation is called an interval graph

**Fact 1** A graph $G$ has a path representation (tolerance $= 1$) if and only if it is an interval graph.

**Fact 2** A graph $G$ has a path representation with tolerance $t$ if and only if it has a path representation with tolerance $t + 1$.

**Path representations**

**Definition 2** A set of vertices in a graph

$$\{x_1, x_2, x_3\}$$

is called an asteroidal triple if there is an $x_i, x_j$-path that $x_k$ is not adjacent to, for $(i, j, k) = (1, 2, 3), (1, 3, 2), (2, 3, 1)$. 
A graph which contains an asteroidal triple is called an asteroidal graph.

Path representations of graphs

The following theorem was proved by Lekkerkerker and Boland in '67 for interval graphs. It was in this paper where the theorem first appeared that every chordal graph has a simplicial vertex (a vertex whose neighbors form a clique) and that there are actually two-non-neighboring simplicial vertices.

**Theorem 1** $G$ is path representable $\iff G$ is chordal and not asteroidal.

Ideal for presentation by a grad student! Opening April 16.

They also provided a complete list of forbidden subgraphs
\[ \text{IV}_n(n + 4 \text{ vertices, } n \geq 2) \]

\[ \text{V}_n(n + 5 \text{ vertices, } n \geq 1) \]
Path representable
\[ \Rightarrow \text{chordal and not asteroidal} \]

We saw that non-chordal graphs cannot be represented by any tree (tolerance = 1).

The problem with asteroidal triples:

Let \( a, b, c \) be three independent vertices represented on a path by disjoint subpaths \( A, B, C \).

The representation of an \( a, c \)-path will have some subpath intersecting \( B \).

Path representable
\[ \Leftarrow \text{chordal and not asteroidal} \]

Definition 3 A vertex \( v \) in a graph \( G \) is said to be strongly simplicial if it is simplicial (its neighborhood is a clique) and the subgraph
< V(G)\S(v) > is connected, where S(v) = N(v) + v.

**Definition 4** If u → P_u is a path representation of G, an end subpath P_v is one that contains a leaf of the host path.

Path representable
⇔ chordal and not asteroidal

**Lemma 1** If G is path representable and has a strongly simplicial vertex v, there is a path representation where the subpath representing v is an end subpath.

They use induction and the two cases:
• Case 1: All simplicial vertices are strongly simplicial

• Case 2: There exists one that isn’t

$k$-ary tree representations

The host tree is a $k$-ary tree. (Rooted, each node has at most $k$ children.)

Reminder $[h, s, t]$ is the class of graphs representable with tolerance $t$ by a host tree with
maximum degree $h$ and subtrees of maximum degree $s$.

We consider classes:

$$[3, 3, 1], \ [3, 3, 2], \ [3, 3, 3], \ [3, 3, 4], \ ...$$

$k$-ary tree representations

McMorris & Scheinerman, '91, showed

**Theorem 2** $G$ is chordal $\iff$ there exists a $T$ representation of $G$ where $\Delta(T) \leq 3$.

So, $[\infty, \infty, 1] = [3, 3, 1] = \text{chordal graphs}$.

Also, Jamison and Mulder, '00 showed that $[3, 3, 1] = [3, 3, 2]$.

**Question 3** What graphs are in the class $[3, 3, 3]$?
It would be nice to have a Lekkerkerker and Boland type characterization with or without a complete list of forbidden subgraphs. Jamison and Mulder have some results toward that end.

\[ K_{3,3}, K_{2,4} \in [3, 3, 3], \ K_{3,4}, K_{2,5} \not\in [3, 3, 3] \]

**Aster representations of graphs**

If for some \( m \), \( G \) is \( H_m \) representable with tolerance \( t \), we say \( G \in \mathcal{H}(t) \).

**Theorem 3** \((E, \ Saadi) \ \mathcal{H}(1) = \mathcal{H}(2) \)
Theorem 4  \((E, \text{Saadi}) \, \mathcal{H}(t) \subseteq \mathcal{H}(t + 1)\)

**Aster representations of cycles**

For \(n \geq 4\), \(C_n \not\in \mathcal{H}(1) = \mathcal{H}(2)\).

**Result 1** \(t = 3, 4, 5\), the maximum \(n\) such that \(C_n \in \mathcal{H}(t)\) is \(n = 3t - 3\).

Notice subtree \((i, j, k)\) adjacent to \((\ell, m, n)\) if \(t - 1 = \min(i, \ell) + \min(j, m) + \min(k, n)\).

**Aster representations of cycles**

**Theorem 5** \((E, \text{Saadi})\) Let \(n\) be the maximum such that \(C_n \in \mathcal{H}(t)\) and \(t \geq 6\) then

\[
\frac{1}{4} t^2 \leq n \leq \frac{3}{10} t^2 + O(t)
\]

**Lemma 2** \(C_n \in \mathcal{H}(t), \, n \geq 4,\) with \(H_m\) representation \(\{S_i\}\), then \(\deg_{S_i}(x) \geq 2\), for all \(i\).
Lemma 3 If $C_n$ has an $H_m$ representation with tolerance $t$, then $C_n$ has an $H_{t-1}$ representation with all subtrees of order $t + 1$.

**Aster representations of cycles**

Consider $G_t = (V_t, E_t)$:

$V_t$ Vertices are subtrees of $H_{t-1}$ which contain $x$ and are size $t+1$. Notation $(i, j, k)$ where $i + j + k = t$.

$E_t$ Edges are between 2 subtrees which intersect in $t$ nodes.

$\{(i, j, k), (d, e, f)\}$ is an edge $\iff$

\[
t - 1 = \min\{i, d\} + \min\{j, e\} + \min\{k, f\}
\]

which is a plane in 3-space. We are only interested in the portion in the octant with all positive coordinates.
Aster representations of cycles
Aster tolerance representations of graphs

Work on characterizing $\mathcal{H}(1) = \mathcal{H}(2)$.

**Definition 5** A set of vertices in a graph

$$\{x_1, x_2, x_3\}$$

is called an **asteroidal triple** if there is an $x_i, x_j$-path that $x_k$ is not adjacent to,

for $(i, j, k) = (1, 2, 3), (1, 3, 2), (2, 3, 1)$.

A set of vertices $\{x_1, x_2, \ldots, x_k\}$ is a **$k$-asteroid** if any 3-element subset is an asteroidal triple.

A graph which contains a $k$-asteroid is called a **$k$-asteroidal graph**.
Work on characterizing $\mathcal{H}(1) = \mathcal{H}(2)$.

**Definition 6** Two 3-asteroids $\{u_1, u_2, u_3\}$ and $\{x_1, x_2, x_3\}$ are said to be **compatible** if all of the following conditions are satisfied and **incompatible** if not.

- Two vertices from $\{u_1, u_2, u_3\}$ create 3-asteroids with each of two vertices from $\{x_1, x_2, x_3\}$.

- There is no cut set $S$ in $G$ such that $\{u_1, u_2, u_3\}$ and $\{x_1, x_2, x_3\}$ are asteroidal triples in two different components of $G \setminus S$.

- For $1 \leq i < j \leq 3$ and $1 \leq k < m \leq 3$, any $u_i, u_j$-path is adjacent to any $v_k, v_m$-path.

**Aster tolerance representations of graphs**
Work on characterizing $\mathcal{H}(1) = \mathcal{H}(2)$.

**Theorem 6** If $G \in \mathcal{H}(1) \Rightarrow G$ is not $k$ asteroidal for $k \geq 4$ and $G$ has no pair of incompatible 3-asteroids.

**Conjecture 1** Assume $G$ is connected, chordal not $k$-asteroidal, and does not contain 2 incompatible 3-asteroids, then $G \in \mathcal{H}(1)$.

**Representations of graphs in Caterpillars**

**Definition 7** A caterpillar is a tree in which a single path (the spine) is incident to (or contains) every edge.

**Representing the Complete bipartite graph in a binary tree**

Since $K_{n,m}$, $n, m \geq 2$ is not chordal, we know that
$K_{2,2} \notin [3,3,1] = [3,3,2]$. So $K_{3,3} \notin [3,3,2]$

However, $K_{3,3} \in [3,3,3]$

One may therefore ask what is the minimum $t = t(n)$ such that $K_{n,n} \in [3,3,t]$.

From above $t(2) = t(3) = 3$

Also $t(4) = 4$. By example, and rather long winded proof.

**Representing the Complete bipartite graph in a binary tree**

For the general case, we prove the following theorem.
Theorem 7 Given $\varepsilon > 0$ there exists an $N_0$ such that $n \geq N_0$ implies

$$t(n) \leq n^\varepsilon$$

and for all $n$, 

$$\frac{\log n}{6} \leq t(n).$$

Also,

Theorem 8

$$K_{n,n} \in [3, 3, t] \Rightarrow K_{n,n} \in [3, 3, t + 1]$$

Areas to explore

♡ Choose a type of subtree
More areas to explore

*The representation number of a graph is the minimum size of a host tree over any type of host tree representation of the graph.

*Allow representing subgraphs to be disconnected, but where the connected components are of a certain type.

♦ Union of stars. Minimum number of stars used for any one vertex is the star number of the graph.
Union of trees. Minimum number trees for any one vertex is the tree number of the graph.