Chapter 6: Section 6.1

Relations.

Definition 1. A relation $R$ on two nonempty sets $A$ and $B$ is a subset $R$ of $A \times B$. If $(a, b) \in R$, we say $aRb$, read “$a$ is related to $b$”. We may say $R$ is a relation in $A \times B$. If $A = B$, we say “$R$ is a relation on $A$” or “$R$ is a relation in $A \times A$.

Definition 2. An equivalence relation is a relation $R$ that is
  • reflexive, $aRa$.
  • transitive, $aRb \land bRc \Rightarrow aRc$ and
  • symmetric, $aRb \Rightarrow bRa$.

Example 1. A partition of a set $S$ forms an equivalence relation $R$ in $S \times S$, by $xRy \iff x$ and $y$ are in the same block of the partition.

Definition 3. A relation $f$ on $A$ and $B$ is a function, denoted $f: A \rightarrow B$ if for all $a \in A$, there exists a unique $b \in B$ such that $(a, b)$ is a member of the relation, denoted $f: a \mapsto b$ or $f(a) = b$.

The set $A$ is the domain and the set $B$ is the codomain.

If $f(a) = b$, we say $b$ is the image of $a$ under $f$ or the value of $f$ at $a$. We also say $a$ is mapped to $b$.

Definition 4. We say a relation $f$ on $A$ and $B$ is well-defined if $f$ satisfies the definition of a function. Two properties must be verified, $\forall a \in A, f(a)$ is defined, and $\forall y_1, y_2 \in A, f(y_1) \neq f(y_2) \Rightarrow y_1 \neq y_2$.

Example 2. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by
  $$f(x) = \begin{cases} 
  \frac{(x+3)^2-9}{x+6}, & x > 0 \\
  x+6, & x < 7 
  \end{cases}$$

if is defined for all $x \in \mathbb{R}$, and for $0 < x < 7$, $\frac{(x+3)^2-9}{x} = x+6$. Therefore, $f$ is a well-defined function.

Definition 5. The identity function on a set $A$ is denoted $id_A$ and is the function that maps $a$ to $a$ for all $a \in A$.

Definition 6. The range of a function $f: A \rightarrow B$ is the set $f(A) = \{f(a) | a \in A\}$. Of course the range is always a subset of the codomain.

Definition 7. A function $f$ is surjective or onto if for all $b \in B$, there exists an $a \in A$ such that $f(a) = b$. Both of the following statements are equivalent to this definition.
  1. The range of $f$ is equal to the codomain.
  2. $f(A) = B$, where $f(A) = \{y \in B : \exists x \in A, f(x) = y\}$. 

Definition 8. A function $f$ is injective or one-to-one if for any $a_1, a_2 \in A$, $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$. Equivalently, for any $a_1, a_2 \in A$, $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$.

Definition 9. A function is a bijection, or a one-to-one correspondence if it is both an injection and surjection.
More from section 6.1 - compositions

Definition 10.

\[
f : A \to B \\
g : B \to C
\]

The composition, \( g \) of \( f \) is denoted \( g \circ f \) and is defined by: \( g \circ f(x) = g(f(x)) \).

Theorem 1. \((6.1.14)\) If \( f \) and \( g \) are both one-to-one, then \( g \circ f \) is one-to-one. If \( f \) and \( g \) are both onto, then \( g \circ f \) is onto.

Examples

Determine if the following functions are one-to-one and/or onto.

(1) \[f : \mathbb{N} \to \mathbb{Z}, f(n) = \begin{cases} -\frac{n-1}{2}, & n \text{ odd} \\ \frac{n}{2}, & n \text{ even} \end{cases}\]

Sol: First we show that \( f \) is one-to-one. Let \( a_1, a_2 \in \mathbb{N} \) and suppose \( f(a_1) = f(a_2) \). Due to the values of \( f(n) \) we break the analysis into 2 cases.

Case 1. \( f(a_1) = f(a_2) \leq 0 \).

In this case, it must be that \( f(a_1) = -\frac{a_1-1}{2} \) and \( f(a_2) = -\frac{a_2-1}{2} \) so we have that \(-\frac{a_1-1}{2} = -\frac{a_2-1}{2}\). Multiplying both sides by \(-2\) gives that \( a_1 - 1 = a_2 - 1 \), so \( a_1 = a_2 \).

Case 2. \( f(a_1) = f(a_2) > 0 \).

In this case, it must be that \( f(a_1) = \frac{a_1}{2} \) and \( f(a_2) = \frac{a_2}{2} \) so we have that \( \frac{a_1}{2} = \frac{a_2}{2} \). Multiplying both sides by \(2\) gives that \( a_1 = a_2 \).

Next, we show that \( f \) is onto. Let \( y \in \mathbb{Z} \). Then again we have two cases.

Case 1. \( y \leq 0 \).

We set \( x = -2y + 1 \). Then, \( x \) is odd, and we have:

\[
f(x) = f(-2y + 1) = -\frac{(-2y + 1) - 1}{2} = -\frac{-2y}{2} = y.
\]

Case 2. \( y > 0 \).

We set \( x = 2y \). Then, \( x \) is even, and we have:

\[
f(x) = f(2y) = \frac{2y}{2} = y.
\]
(2) \[ f : \mathbb{R}^2 \rightarrow \mathbb{R}^2, f(x,y) = (x-y, x+y) \]

**Sol:** First we show that \( f \) is one-to-one. Let \((x_1, y_1), (x_2, y_2) \in \mathbb{R}^2\) and assume \( f(x_1, y_1) = f(x_2, y_2) \). This implies that \((x_1 - y_1, x_1 + y_1) = (x_2 - y_2, x_2 + y_2)\). Using the fact that like coordinates are equal, we have that \( x_1 - y_1 = x_2 - y_2 \) and \( x_1 + y_1 = x_2 + y_2 \). Adding the two equations together gives, \( 2x_1 = 2x_2 \) so \( x_1 = x_2 \). By substitution we see that \( y_1 = y_2 \).

Now we show that \( f \) is onto. Let \((a, b)\) be an arbitrary element of \( \mathbb{R}^2 \). Set \( x = \frac{a+b}{2} \) and \( y = \frac{b-a}{2} \). Then
\[
 f(x, y) = \left( \frac{a+b}{2} - \frac{b-a}{2}, \frac{a+b}{2} + \frac{b-a}{2} \right) = \left( \frac{a + b - b + a}{2}, \frac{a + b + b - a}{2} \right) = (a, b)
\]

**Bounded.** We say a function on \( \mathbb{R} \) is **bounded** if \( \exists M \in \mathbb{R} \) such that \( \forall x \in \mathbb{R}, f(x) \leq M \).

**Increasing and decreasing.** Let \( f \) be a function on \( \mathbb{R} \). We say
- \( f \) is increasing if \( \forall x, y \in \mathbb{R} \) such that \( x < y \), we have \( f(x) \leq f(y) \).
- \( f \) is strictly increasing if \( \forall x, y \in \mathbb{R} \) such that \( x < y \), we have \( f(x) < f(y) \).
- \( f \) is decreasing if \( \forall x, y \in \mathbb{R} \) such that \( x < y \), we have \( f(x) \geq f(y) \).
- \( f \) is strictly decreasing if \( \forall x, y \in \mathbb{R} \) such that \( x < y \), we have \( f(x) > f(y) \).