**Chapter 2**

**Defn 1.** (p. 65) Let $V$ and $W$ be vector spaces (over $F$). We call a function $T : V \to W$ a linear transformation form $V$ to $W$, if, for all $x, y \in V$ and $c \in F$, we have

(a) $T(x + y) = T(x) + T(y)$ and
(b) $T(cx) = cT(x)$.

**Fact 1.** (p. 65)

1. If $T$ is linear, then $T(0) = 0$. Note that the first $0$ is in $V$ and the second one is in $W$.
2. $T$ is linear if and only if $T(cx + y) = cT(x) + T(y)$ for all $x, y \in V$ and $c \in F$.
3. If $T$ is linear, then $T(x - y) = T(x) - T(y)$ for all $x, y \in V$.
4. $T$ is linear if and only if, for $x_1, x_2, \ldots, x_n \in V$ and $a_1, a_2, \ldots, a_n \in F$, we have

$$T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i).$$

**Proof.**

1. We know $0 \cdot x = 0$ for all vectors $x$. Hence, $0 = 0 \cdot T(x) = T(0 \cdot x) = T(0)$.
2. Assume $T$ is linear and $c \in F$ and $x, y \in V$, then $T(cx) = cT(x)$ and $T(cx + y) = T(cx) + T(y) = cT(x) + T(y)$.

   Assume $T(x + y) = T(x) + T(y)$ for all $x, y \in V$. Let $a \in F$ and $x, y \in V$. Then $T(ax) = T(ax + 0) = aT(x) + T(0) = aT(x)$ and $T(x + y) = T(1 \cdot x + y) = 1 \cdot T(x) + T(y) = T(x) + T(y)$.
3. To start with, the definition of subtraction is: $x - y = x + (-y)$. We also wish to show: $\forall x \in V, -1 \cdot x = -x$. Given $0 \cdot x = 0$ which we proved earlier, we have

$$\begin{align*}
(1 + (-1))x &= 0 \\
1 \cdot x + (-1) \cdot x &= 0
\end{align*}$$

Since the additive inverse in unique, we conclude that $(-1) \cdot x = -x$.

Assume $T$ is linear. Then $T(x - y) = T(x + (-1) \cdot y) = T(x) + (-1)T(y) = T(x) + T(-y) = T(x) - T(y)$ for all $x, y \in V$.
4. ($\Rightarrow$) Let $n = 2$, $a_1 = c$, and $a_2 = 1$. We have that $T(cx_1 + x_2) = cT(x_1) + T(x_2)$.

($\Rightarrow$) This is done by induction. Using the definition of linear transformation, we get the base case ($n = 2$). That is,

$$T(a_1 x_1 + a_2 x_2) = T(a_1 x_1) + T(a_2 x_2) = a_1 T(x_1) + a_2 T(x_2).$$

Now assume that $n \geq 2$ and for all $x_1, x_2, \ldots, x_n \in V$ and $a_1, a_2, \ldots, a_n \in F$, we have

$$T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i).$$
Then for all \(x_1, x_2, \ldots, x_n, x_{n+1} \in V\) and \(a_1, a_2, \ldots, a_n, a_{n+1} \in F\), we have
\[
T\left(\sum_{i=1}^{n+1} a_i x_i\right) = T\left(\sum_{i=1}^{n} a_i x_i + a_{n+1} x_{n+1}\right)
\]
\[
= T\left(\sum_{i=1}^{n} a_i x_i\right) + T\left(a_{n+1} x_{n+1}\right)
\]
\[
= \sum_{i=1}^{n} a_i T(x_i) + a_{n+1} T(x_{n+1})
\]
\[
= \sum_{i=1}^{n+1} a_i T(x_i).
\]

\[\blacksquare\]

**Defn 2.** (p. 67) Let \(V\) and \(W\) be vector spaces, and let \(T : V \rightarrow W\) be linear. We define the null space (or kernel) \(N(T)\) of \(T\) to be the set of all vectors \(x\) in \(V\) such that \(T(x) = 0\); \(N(T) = \{x \in V : T(x) = 0\}\).

We define the range (or image) \(R(T)\) of \(T\) to be the subset \(W\) consisting of all images (under \(T\)) of vectors in \(V\); that is, \(R(T) = \{T(x) : x \in V\}\).

**Theorem 2.1.** Let \(V\) and \(W\) be vector spaces and \(T : V \rightarrow W\) be linear. Then \(N(T)\) and \(R(T)\) are subspaces of \(V\) and \(W\), respectively.

**Proof.** To prove that \(N(T)\) is a subspace of \(V\), we let \(x, y \in N(T)\) and \(c \in F\). We will show that \(0 \in N(T)\) and \(cx + y \in N(T)\). We know that \(T(0) = 0\). So, \(0 \in N(T)\). We have that \(T(x) = 0\) and \(T(y) = 0\) so \(T(cx + y) = c \cdot T(x) + T(y) = c \cdot 0 + 0 = c \cdot 0 = 0\). (The last equality is a fact that can be proved from the axioms of vector space.

To prove that \(R(T)\) is a subspace of \(W\), assume \(T(x), T(y) \in R(T)\), for some \(x, y \in V\) and \(c \in F\). Then, \(cT(x) + T(y) = T(cx + y)\), where \(cx + y \in V\), since \(T\) is a linear transformation and so, \(cT(x) + T(y) \in R(T)\). Also, since \(0 \in V\) and \(0 = T(0)\), we know that \(0 \in R(T)\). \[\blacksquare\]

Let \(A\) be a set of vectors in \(V\). Then \(T(A) = \{T(x) : x \in A\}\).

**Theorem 2.2.** Let \(V\) and \(W\) be vector spaces, and let \(T : V \rightarrow W\) be linear. If \(\beta = \{v_1, v_2, \ldots, v_n\}\) is a basis for \(V\), then \(R(T) = \text{span}(T(\beta))\).

**Proof.** \(T(\beta) = \{T(v_1), T(v_2), \ldots, T(v_n)\}\). We will show \(R(T) = \text{span}(T(\beta))\). We showed that \(R(T)\) is a subspace of \(W\). We know that \(T(\beta) \subseteq R(T)\). So, by Theorem 1.5, \(\text{span}(T(\beta)) \subseteq R(T)\).

Let \(x \in R(T)\). Then there is some \(x' \in V\) such that \(T(x') = x\). \(\beta\) is a basis of \(V\), so there exist scalars \(a_1, a_2, \ldots, a_n\) such that \(x' = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n\).

\[
\begin{align*}
x &= T(x') \\
&= T(a_1 x_1 + a_2 x_2 + \cdots + a_n x_n) \\
&= a_1 T(x_1) + a_2 T(x_2) + \cdots + a_n T(x_n)
\end{align*}
\]

which is in \(\text{span}(T(\beta))\).

Thus \(R(T) \subseteq \text{span}(T(\beta))\). \[\blacksquare\]
**Defn 3.** (p. 69) Let $V$ and $W$ be vector spaces, and let $T : V \to W$ be linear. If $N(T)$ and $R(T)$ are finite-dimensional, then we define the nullity of $T$, denoted $\text{nullity}(T)$, and the rank of $T$, denoted $\text{rank}(T)$, to be the dimensions of $N(T)$ and $R(T)$, respectively.

**Theorem 2.3. (Dimension Theorem).** Let $V$ and $W$ be vector spaces, and let $T : V \to W$ be linear. If $V$ is finite-dimensional, then $\text{nullity}(T) + \text{rank}(T) = \text{dim}(V)$.

**Proof.** Suppose $\text{nullity}(T) = k$ and $\beta = \{\mathbf{n}_1, \ldots, \mathbf{n}_k\}$ is a basis of $N(T)$. Since $\beta$ is linearly independent in $V$, then $\beta$ extends to a basis of $V$,

$$\beta' = \{\mathbf{n}_1, \ldots, \mathbf{n}_k, \mathbf{x}_{k+1}, \ldots, \mathbf{x}_n\},$$

where, by Corollary 2c. to Theorem 1.10, $\forall i, \mathbf{x}_i \notin N(T)$. Notice that possibly $\beta = \emptyset$.

It suffices to show:

$$\beta'' = \{T(\mathbf{x}_{k+1}), \ldots, T(\mathbf{x}_n)\}$$

is a basis for $R(T)$.

By Theorem 2.2, $T(\beta')$ spans $R(T)$. But $T(\beta'') = \{0\} \cup \beta''$. So $\beta''$ spans $R(T)$. Suppose $\beta''$ is not linearly independent. Then there exist $c_1, c_2, \ldots, c_n \in F$, not all 0 such that

$$0 = c_1 T(\mathbf{x}_{k+1}) + \cdots + c_n T(\mathbf{x}_n).$$

Then

$$0 = T(c_1 \mathbf{x}_{k+1} + \cdots + c_n \mathbf{x}_n)$$

and

$$c_1 \mathbf{x}_{k+1} + \cdots + c_n \mathbf{x}_n \in N(T).$$

But then

$$c_1 \mathbf{x}_{k+1} + \cdots + c_n \mathbf{x}_n = a_1 \mathbf{n}_1 + a_2 \mathbf{n}_2 + \cdots + a_k \mathbf{n}_k$$

for some scalars $a_1, a_2, \ldots, a_k$, since $\beta = \{\mathbf{n}_1, \ldots, \mathbf{n}_k\}$ is a basis of $N(T)$. Then

$$0 = a_1 \mathbf{n}_1 + a_2 \mathbf{n}_2 + \cdots + a_k \mathbf{n}_k - c_1 \mathbf{x}_{k+1} - \cdots - c_n \mathbf{x}_n.$$ 

The $c_i$'s are not all zero, so this contradicts that $\beta'$ is a linearly independent set. ■

**Theorem 2.4.** Let $V$ and $W$ be vector spaces, and let $T : V \to W$ be linear. Then $T$ is one-to-one if and only if $N(T) = \{0\}$.

**Proof.** Suppose $T$ is one-to-one. Let $\mathbf{x} \in N(T)$. Then $T(\mathbf{x}) = 0$ also $T(0) = 0$. $T$ being one-to-one implies that $\mathbf{x} = 0$. Therefore, $N(T) \subseteq \{0\}$. It is clear that $\{0\} \subseteq N(T)$. Therefore, $N(T) = \{0\}$.

Suppose $N(T) = \{0\}$. Suppose for $\mathbf{x}, \mathbf{y} \in V, T(\mathbf{x}) = T(\mathbf{y})$. We have that $T(\mathbf{x}) - T(\mathbf{y}) = 0$. $T(\mathbf{x} - \mathbf{y}) = 0$. Then it must be that $\mathbf{x} - \mathbf{y} = 0$. This implies that $-\mathbf{y}$ is the additive inverse of $\mathbf{x}$ and $\mathbf{x}$ is the additive inverse of $-\mathbf{y}$. But also $\mathbf{y}$ is the additive inverse of $-\mathbf{y}$. By uniqueness of additive inverses we have that $\mathbf{x} = \mathbf{y}$. Thus, $T$ is one-to-one. ■

**Theorem 2.5.** Let $V$ and $W$ be vector spaces of equal (finite) dimension, and let $T : V \to W$ be linear. Then the following are equivalent.

(a) $T$ is one-to-one.

(b) $T$ is onto.

(c) $\text{rank}(T) = \text{dim}(V)$.
Proof. Assume \( \dim(V) = \dim(W) = n \).

(a) \( \Rightarrow \) (b) : To prove \( T \) is onto, we show that \( R(T) = W \). Since \( T \) is one-to-one, we know \( N(T) = \{0\} \) and so \( \nullity(T) = 0 \). By Theorem 2.3, \( \nullity(T) + \rank(T) = \dim(V) \). So we have \( \dim(W) = n = \rank(T) \). By Theorem 1.11, \( R(T) = W \).

(b) \( \Rightarrow \) (c) : \( T \) is onto implies that \( R(T) = W \). So \( \rank(T) = \dim(W) = \dim(V) \).

(c) \( \Rightarrow \) (a) : If we assume \( \rank(T) = \dim(V) \), By Theorem 2.3 again, we have that \( \nullity(T) = 0 \). But then we know \( N(T) = \{0\} \). By Theorem 2.4, \( T \) is one-to-one.

\[ \textbf{Theorem 2.6.} \] Let \( V \) and \( W \) be vector spaces over \( F \), and suppose that \( \beta = \{v_1, v_2, \ldots, v_n\} \) is a basis for \( V \). For \( w_1, w_2, \ldots, w_n \) in \( W \), there exists exactly one linear transformation \( T : V \to W \) such that \( T(v_i) = w_i \) for \( i = 1, 2, \ldots, n \).

Proof. Define a function \( T : V \to W \) by \( T(v_i) = w_i \) for \( i = 1, 2, \ldots, n \) and since \( \beta \) is a basis, we can express every vector in \( V \) as a linear combination of vectors in \( \beta \). For \( x \in V \), \( x = a_1v_1 + a_2v_2 + \cdots + a_nv_n \), we define \( T(x) = a_1w_1 + a_2w_2 + \cdots + a_2w_n \).

We will show that \( T \) is linear. Let \( c \in F \) and \( x, y \in V \). Assume \( x = a_1v_1 + a_2v_2 + \cdots + a_nv_n \) and \( y = b_1v_1 + b_2v_2 + \cdots + b_nv_n \). Then

\[
\begin{align*}
  cx + y &= \sum_{i=1}^{n} (ca_i + b_i)v_i \\
  T(cx + y) &= T\left(\sum_{i=1}^{n} (ca_i + b_i)v_i\right) \\
  &= \sum_{i=1}^{n} (ca_i + b_i)w_i \\
  &= c\sum_{i=1}^{n} a_iw_i + \sum_{i=1}^{n} b_iw_i \\
  &= cT(x) + T(y)
\end{align*}
\]

Thus, \( T \) is linear.

To show it is unique, we need to show that if \( F : V \to W \) is a linear transformation such that \( F(v_i) = w_i \) for all \( i \in [n] \), then for all \( x \in V \), \( F(x) = T(x) \).

Assume \( x = a_1v_1 + a_2v_2 + \cdots + a_nv_n \). Then

\[
\begin{align*}
  T(x) &= a_1w_1 + a_2w_2 + \cdots + a_nw_n \\
  &= a_1F(v_1) + a_2F(v_2) + \cdots + a_nF(v_n) \\
  &= F(x)
\end{align*}
\]

\[ \text{Cor 1.} \] Let \( V \) and \( W \) be vector spaces, and suppose that \( V \) has a finite basis \( \{v_1, v_2, \ldots, v_n\} \).
If \( U, T : V \to W \) are linear and \( U(v_i) = T(v_i) \) for \( i = 1, 2, \ldots, n \), then \( U = T \).

\[ \text{Defn 4. (p. 80)} \] Given a finite dimensional vector space \( V \), an ordered basis is a basis where the vectors are listed in a particular order, indicated by subscripts. For \( F^n \), we have \( \{e_1, e_2, \ldots, e_n\} \) where \( e_i \) has a 1 in position \( i \) and zeros elsewhere. This is known as the standard ordered basis and \( e_i \) is the \( i^{\text{th}} \) characteristic vector. Let \( \beta = \{u_1, u_2, \ldots, u_n\} \) be an ordered basis for a finite-dimensional vector space \( V \). For \( x \in V \), let \( a_1, a_2, \ldots, a_n \) be the
unique scalars such that
\[ x = \sum_{i=1}^{n} a_i u_i. \]

We define the coordinate vector of \( x \) relative to \( \beta \), denoted \([x]_\beta\), by
\[ [x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}. \]

**Defn 5.** (p. 80) Let \( V \) and \( W \) be finite-dimensional vector spaces with ordered bases \( \beta = \{v_1, v_2, \ldots, v_n\} \) and \( \gamma = \{w_1, w_2, \ldots, w_n\} \), respectively. Let \( T : V \to W \) be linear. Then for each \( j, 1 \leq j \leq n \), there exists unique scalars \( a_{i,j} \in F \), \( 1 \leq i \leq m \), such that
\[ T(v_j) = \sum_{i=1}^{m} t_{i,j} w_i \quad 1 \leq j \leq n. \]

So that
\[ [T(v_j)]_\gamma = \begin{pmatrix} t_{1,j} \\ t_{2,j} \\ \vdots \\ t_{m,j} \end{pmatrix}. \]

We call the \( m \times n \) matrix \( A \) defined by \( A_{i,j} = a_{i,j} \) the matrix representation of \( T \) in the ordered bases \( \beta \) and \( \gamma \) and write \( A = [T]_\beta^\gamma \). If \( V = W \) and \( \beta = \gamma \), then we write \( A = [T]_\beta \).

**Fact 2.** \([a_1x_1 + a_2x_2 + \cdots + a_nx_n]_B = a_1[x_1]_B + a_2[x_2]_B + \cdots + a_n[x_n]_B\)

**Proof.** Let \( \beta = \{v_1, v_2, \ldots, v_n\} \). Say \( x_j = b_{j,1}v_1 + b_{j,2}v_2 + \cdots + b_{j,n}v_n \).

The LHS: \[ a_1x_1 + a_2x_2 + \cdots + a_nx_n \]
\[ = a_1(b_{1,1}v_1 + b_{1,2}v_2 + \cdots + b_{1,n}v_n) + a_2(b_{2,1}v_1 + b_{2,2}v_2 + \cdots + b_{2,n}v_n) + \cdots + a_n(b_{n,1}v_1 + b_{n,2}v_2 + \cdots + b_{n,n}v_n) \]
\[ = (a_1b_{1,1} + a_2b_{2,1} + \cdots + a_nb_{n,1})v_1 + (a_1b_{1,2} + a_2b_{2,2} + \cdots + a_nb_{n,2})v_2 + \cdots + (a_1b_{1,n} + a_2b_{2,n} + \cdots + a_nb_{n,n})v_n \]

The \( i^{th} \) position of the LHS is the coefficient \( c_i \) of \( v_i \) in:
\[ a_1x_1 + a_2x_2 + \cdots + a_nx_n = c_1v_1 + c_2v_2 + \cdots + c_nv_n \]
And so, \( c_i = a_1b_{1,i} + a_2b_{2,i} + \cdots + a_nb_{n,i} \).

The RHS: The \( i^{th} \) position of the RHS is the sum of the \( i^{th} \) position of each \( a_j[x_j]_\beta \), which is the coefficient of \( v_i \) in \( a_jx_j = a_j(b_{j,1}v_1 + b_{j,2}v_2 + \cdots + b_{j,n}v_n) \) and thus, \( a_jb_{j,i} \). We have the \( i^{th} \) position of the RHS is \( a_1b_{1,i} + a_2b_{2,i} + \cdots + a_nb_{n,i} \). \qed

**Theorem 6.** (p. 80) Let \( V \) be a finite-dimensional vector space with ordered basis \( \beta = \{v_1, v_2, \ldots, v_n\} \). Let \( T : V \to V \) be linear. Then there exists a unique matrix \( A = [T]_\beta^\beta \) such that
\[ T(v) = \sum_{i=1}^{m} t_{i,j} v_i \quad 1 \leq j \leq n. \]

So that
\[ [T(v)]_\beta = \begin{pmatrix} t_{1,j} \\ t_{2,j} \\ \vdots \\ t_{m,j} \end{pmatrix}. \]

We call \( A \) the matrix representation of \( T \) in the ordered basis \( \beta \) and write \( A = [T]_\beta^\beta \). If \( V = W \) and \( \beta = \gamma \), then we write \( A = [T]_\beta \).
Fact 2. (a) Let $V$ and $W$ be finite dimensional vector spaces with bases, $\beta$ and $\gamma$, respectively. 
$[T]_{\beta}^\gamma(x) = [T(x)]_{\gamma}$.

Proof. Let $\beta = \{v_1, v_2, \ldots, v_n\}$. Let $x = a_1v_1 + a_2v_2 + \cdots + a_nv_n$. Then

$$
[T(x)]_{\gamma} = [T(a_1v_1 + a_2v_2 + \cdots + a_nv_n)]_{\gamma} = [\alpha_1T(v_1) + \alpha_2T(v_2) + \cdots + \alpha_nT(v_n)]_{\gamma} = [\alpha_1T(v_1)]_{\gamma} + [\alpha_2T(v_2)]_{\gamma} + \cdots + [\alpha_nT(v_n)]_{\gamma}\text{by fact 2} 
$$

$$
= \begin{bmatrix}
t_{1,1} & \cdots & t_{1,n} \\
\vdots & & \ddots \\
t_{n,1} & \cdots & t_{n,n}
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\vdots \\
\alpha_n
\end{bmatrix}
= [T]_{\beta}^\gamma(x).
$$

Fact 3. Let $V$ be a vector space of dimension $n$ with basis $\beta$. $\{x_1, x_2, \ldots, x_k\}$ is linearly independent in $V$ if and only if $\{[x_1]_{\beta}, [x_2]_{\beta}, \ldots, [x_k]_{\beta}\}$ is linearly independent in $F^n$.

Proof.

$\{x_1, x_2, \ldots, x_k\}$ is linearly dependent

$\iff \exists a_1, a_2, \ldots, a_k$, not all zero, such that $a_1x_1 + a_2x_2 + \cdots + a_kx_k = 0$

$\iff \exists a_1, a_2, \ldots, a_k$, not all zero, such that $[a_1x_1 + a_2x_2 + \cdots + a_kx_k]_{\beta} = [0]_{\beta} = 0$

since the only way to represent $0(\in V)$ in any basis is $0(\in F^n)$

$\iff \exists a_1, a_2, \ldots, a_k$, not all zero, such that $a_1[x_1]_{\beta} + a_2[x_2]_{\beta} + \cdots + a_k[x_k]_{\beta} = 0$

$\iff \{[x_1]_{\beta}, [x_2]_{\beta}, \ldots, [x_k]_{\beta}\}$ is a linearly dependent set

Defn 6. (p. 99) Let $V$ and $W$ be vector spaces, and let $T : V \rightarrow W$ be linear. A function $U : W \rightarrow V$ is said to be an inverse of $T$ if $TU = I_W$ and $UT = I_V$. If $T$ has an inverse, then $T$ is said to be invertible.

Fact 4. $v$ and $W$ are vector spaces with $T : V \rightarrow W$. If $T$ is invertible, then the inverse of $T$ is unique.

Proof. Suppose $U : W \rightarrow V$ is such that $TU = I_W$ and $UT = I_V$. $X : W \rightarrow V$ is such that $TX = I_W$ and $XT = I_V$.

To show $U = X$, we must show that $\forall w \in W$, $U(w) = X(w)$. We know $I_W(w) = w$ and $I_V(X(w)) = X(w)$.

$$
U(w) = U(I_W(w)) = UTX(w) = I_VX(w) = X(w).
$$

Defn 7. We denote the inverse of $T$ by $T^{-1}$.

Theorem 2.17. Let $V$ and $W$ be vector spaces, and let $T : V \rightarrow W$ be linear and invertible. Then $T^{-1} : W \rightarrow V$ is linear.
Proof. By the definition of invertible, we have: \( \forall \mathbf{w} \in W, T(T^{-1}(\mathbf{w})) = \mathbf{w} \). Let \( \mathbf{x}, \mathbf{y} \in W \).

Then,

\[
T^{-1}(c\mathbf{x} + \mathbf{y}) = T^{-1}(cT(T^{-1}(\mathbf{x})) + T(T^{-1}(\mathbf{y})) = T^{-1}(cT(T^{-1}(\mathbf{x})) + T^{-1}(\mathbf{y})) = T^{-1}(cT^{-1}(\mathbf{x}) + T^{-1}(\mathbf{y})) = T^{-1}(\mathbf{x}) + T^{-1}(\mathbf{y})
\]

Fact 5. If \( T \) is a linear transformation between vector spaces of equal (finite) dimension, then the conditions of being a.) invertible, b.) one-to-one, and c.) onto are all equivalent.

Proof. We start with a) \( \Rightarrow \) c). We have \( TT^{-1} = \mathbf{I}_V \). To show onto, let \( \mathbf{v} \in V \) then for \( \mathbf{x} = T^{-1}(\mathbf{v}) \), we have: \( T(\mathbf{x}) = T(T^{-1}(\mathbf{v})) = \mathbf{I}_V(\mathbf{v}) = \mathbf{v} \). Therefore, \( T \) is onto.

By Theorem 2.5, we have b) \( \iff \) c).

We will show b) \( \Rightarrow \) a).

Let \( \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \) be a basis of \( V \). Then \( \{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)\} \) spans \( R(T) \) by Theorem 2.2. One of the corollaries to the Replacement Theorem implies that \( \{\mathbf{x}_1 = T(\mathbf{v}_1), \mathbf{x}_2 = T(\mathbf{v}_2), \ldots, \mathbf{x}_n = T(\mathbf{v}_n)\} \) is a basis for \( V \).

We define \( U: V \to V \) by \( \forall \mathbf{v}, U(\mathbf{u}) = \mathbf{v} \). By Theorem 2.6, this is a well-defined linear transformation.

Let \( \mathbf{x} \in V \), \( \mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n \) for some scalars \( a_1, a_2, \ldots, a_n \).

\[
TU(\mathbf{x}) = TU(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n) = U(a_1T(\mathbf{v}_1) + a_2T(\mathbf{v}_2) + \cdots + a_nT(\mathbf{v}_n)) = U(a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_n\mathbf{x}_n) = a_1U(\mathbf{x}_1) + a_2U(\mathbf{x}_2) + \cdots + a_nU(\mathbf{x}_n) = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{x}
\]

Let \( \mathbf{x} \in V \), \( \mathbf{x} = b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + \cdots + b_n\mathbf{x}_n \) for some scalars \( b_1, b_2, \ldots, b_n \).

\[
TU(\mathbf{x}) = TU(b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + \cdots + b_n\mathbf{x}_n) = T(b_1U(\mathbf{x}_1) + b_2T(\mathbf{x}_2) + \cdots + b_nT(\mathbf{x}_n)) = T(b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n) = b_1T(\mathbf{v}_1) + b_2T(\mathbf{v}_2) + \cdots + b_nT(\mathbf{v}_n) = b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + \cdots + b_n\mathbf{x}_n = \mathbf{x}
\]

Defn 8. (p. 100) Let \( A \) be an \( n \times n \) matrix. Then \( A \) is invertible if there exits an \( n \times n \) matrix \( B \) such that \( AB = BA = I_n \).

Fact 6. If \( A \) is invertible, then the inverse of \( A \) is unique.

Proof. \( A \in M_n(\mathbb{F}) \) and \( AB = BA = I_n \). Suppose \( AC = CA = I_n \). Then we have \( B = BI_n = BAC = I_nC = C \).
Defn 9. We denote the inverse of $A$ by $A^{-1}$.

Lemma 1. (p. 101) Let $T : V \to W$ be an invertible linear transformation from $V$ to $W$. Then $V$ is finite-dimensional if and only if $W$ is finite-dimensional. In this case, $\dim(V) = \dim(W)$.

Proof. ($\Rightarrow$) $T : V \to W$ is invertible. Then $TT^{-1} = I_W$ and $T^{-1}T = I_V$. Assume $\dim V = n$ and $\{v_1, v_2, \ldots, v_n\}$ is a basis of $V$.

We know $T$ is onto since, if $y \in W$ then for $x = T^{-1}(y)$, we have $T(x) = y$.

We know $R(T) = W$ since $T$ is onto and $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ spans $W$, by Theorem 2.2.

So $W$ is finite dimensional and $\dim W \leq n = \dim V$.

($\Leftarrow$) Assume $\dim W = m$. $T^{-1} : W \to V$ is invertible. Applying the same argument as in the last case, we obtain that $V$ is finite dimensional and $\dim V \leq \dim W$.

We see that when either of $V$ or $W$ is finite dimensional then so is the other and so $\dim W \leq \dim V$ and $\dim V \leq \dim W$ imply that $\dim V = \dim W$.

Theorem 2.18. Let $V$ and $W$ be finite-dimensional vector spaces with ordered bases $\beta$ and $\gamma$, respectively. Let $T : V \to W$ be linear. Then $T$ is invertible if and only if $[T]_\beta^\gamma$ is invertible. Furthermore, $[T^{-1}]_\beta^\gamma = ([T]_\beta^\gamma)^{-1}$.

Proof. Assume $T$ is invertible. Let $\dim V = n$ and $\dim W = m$. By the lemma, $n = m$. Let $\beta = \{v_1, v_2, \ldots, v_n\}$ and $\gamma = \{w_1, w_2, \ldots, w_n\}$.

We have $T^{-1}$ such that $TT^{-1} = I_W$ and $T^{-1}T = I_V$.

We will show that $[T^{-1}]_\gamma^\beta$ is the inverse of $[T]_\beta^\gamma$.

We will show the following 2 matrix equations:

$$[T]_\beta^\gamma [T^{-1}]_\gamma^\beta = I_n$$

(6)

$$[T^{-1}]_\beta^\gamma [T]_\beta^\gamma = I_n$$

(7)

To prove (1) we will take the approach of showing that the $i^{th}$ column of $[T]_\beta^\gamma [T^{-1}]_\gamma^\beta$ is $e_i$, the $i^{th}$ characteristic vector.

Notice that for any matrix $A$, $Ae_i$ is the $i^{th}$ column of $A$.

Consider $[T]_\beta^\gamma [T^{-1}]_\gamma^\beta e_i$. By repeated use of Fact 2a, we have:

$$[T]_\beta^\gamma [T^{-1}]_\gamma^\beta e_i = [T]_\beta^\gamma [T^{-1}]_\gamma^\beta [w_1]_\gamma = [T]_\beta^\gamma [T^{-1}(w_1)]_\beta = [TT^{-1}(w_1)]_\gamma = [w_1]_\gamma = e_i$$

To prove (7) the proof is very similar. We will show that the $i^{th}$ column of $[T^{-1}]_\beta^\gamma [T]_\beta^\gamma$ is $e_i$, the $i^{th}$. 

Consider \( [T^{-1}]_\beta^\gamma [T]_\beta^\gamma \mathbf{e}_1 \). By repeated use of Fact 2a, we have:

\[
[T^{-1}]_\beta^\gamma [T]_\beta^\gamma \mathbf{e}_1 = [T^{-1}]_\beta^\gamma [T(\mathbf{v}_1)]_\gamma \\
= [T^{-1}]_\gamma^\beta [T(\mathbf{v}_1)]_\beta \\
= [\mathbf{v}_1]_\beta \\
= \mathbf{e}_1
\]

Thus we have \((\Rightarrow)\) and the Furthermore part of the statement.

Now we assume \( A = [T]_\beta^\gamma \) is invertible. Call its inverse \( A^{-1} \). We know that it is square, thus \( n = m \). Let \( \beta = \{v_1, v_2, \ldots, v_n\} \) and \( \gamma = \{w_1, w_2, \ldots, w_n\} \). Notice that the \( i^{th} \) column of \( A \) is \( A_i = (t_{1,i}, t_{2,i}, \ldots, t_{n,i}) \) where \( T(\mathbf{v}_i) = t_{1,i}w_1 + t_{2,i}w_2 + \ldots + t_{n,i}w_n \).

To show \( T \) is invertible, we will define a function \( U \) and prove that it is the inverse of \( T \).

Let the \( i^{th} \) column of \( A^{-1} \) be \( C_i = (c_{1,i}, c_{2,i}, \ldots, c_{n,i}) \). We define \( U : W \to V \) by for all \( i \in [n] \),

\[
U(w_i) = c_{1,i}v_1 + c_{2,i}v_2 + \cdots + c_{n,i}v_n
\]

Since we have defined \( U \) for the basis vectors \( \gamma \), we know from Theorem 2.6 that \( U \) is a linear transformation.

We wish to show that \( TU \) is the identity transformation in \( W \) and \( UT \) is the identity transformation in \( V \). So, we show:

1. \( TU(x) = x, \forall x \in W \) and
2. \( UT(x) = x, \forall x \in V \).

Starting with (1.). First lets see why, \( \forall i \in [n], T(U(w_i)) = w_i \).

\[
T(U(w_i)) = T(c_{1,i}v_1 + c_{2,i}v_2 + \cdots + c_{n,i}v_n) \\
= c_{1,i}(t_{1,1}w_1 + t_{2,1}w_2 + \cdots + t_{n,1}w_n) \\
+ c_{2,i}(t_{1,2}w_1 + t_{2,2}w_2 + \cdots + t_{n,2}w_n) \\
+ \cdots + c_{n,i}(t_{1,n}w_1 + t_{2,n}w_2 + \cdots + t_{n,n}w_n)
\]

Gathering coefficients of \( w_1, w_2, \ldots, w_n \), we have:

\[
T(U(w_i)) = (c_{1,i}t_{1,1} + c_{2,i}t_{1,2} + \cdots + c_{n,i}t_{1,n})w_1 \\
+ (c_{1,i}t_{2,1} + c_{2,i}t_{2,2} + \cdots + c_{n,i}t_{2,n})w_2 \\
+ \cdots + (c_{1,i}t_{n,1} + c_{2,i}t_{n,2} + \cdots + c_{n,i}t_{n,n})w_n
\]

We see that the coefficient of \( w_j \) in the above expression is Row \( j \) of \( A \) dot Column \( i \) of \( A^{-1} \), which is always equal to 1 if and only if \( i = j \).

Thus we have that \( T(U(w_i)) = w_i \).

Now we see that for \( x \in W \) then \( x = b_1w_1 + b_2w_2 + \cdots + b_nw_n \) for some scalars \( b_1, b_2, \ldots, b_n \).

\[
TU(x) = TU(b_1w_1 + b_2w_2 + \cdots + b_nw_n) \\
= T(b_1U(w_1) + b_2U(w_2) + \cdots + b_nU(w_n)) \\
= b_1T(U(w_1)) + b_2T(U(w_2)) + \cdots + b_nT(U(w_n)) \\
= b_1w_1 + b_2w_2 + \cdots + b_nw_n \\
= x
\]
Similarly, $UT(x) = x$ for all $x$ in $V$.

**Cor 1.** Let $V$ be a finite-dimensional vector space with an ordered basis $\beta$, and let $T : V \to V$ be linear. Then $T$ is invertible if and only if $[T]_\beta$ is invertible. Furthermore, $[T^{-1}]_\beta = ([T]_\beta)^{-1}$.

**Proof.** Clear.

**Defn 10.** Let $A \in M_{m \times n}(F)$. Then the function $L_A : F^n \to F^m$ where for $x \in F^n$, $L_A(x) = Ax$ and is called left-multiplication by $A$.

**Fact 6.** (a) $L_A$ is a linear transformation and for $\beta = \{e_1, e_2, \ldots, e_n\}$, $[L_A]_\beta = A$.

**Proof.** We showed in class.

**Cor 2.** Let $A$ be an $n \times n$ matrix. Then $A$ is invertible if and only if $L_A$ is invertible. Furthermore, $(L_A)^{-1} = L_{A^{-1}}$.

**Proof.** Let $V = F^n$. Then $L_A : V \to V$. Apply Corollary 1 with $\beta = \{e_1, e_2, \ldots, e_n\}$ and $[L_A]_\beta = A$, we have:

$L_A$ is invertible if and only if $A$ is invertible.

The furthermore part of this corollary says that $(L_A)^{-1}$ is left multiplication by $A^{-1}$.

We will show $L_AL_{A^{-1}} = L_{A^{-1}}L_A = I_V$. So we must show $L_A L_{A^{-1}}(x) = x, \forall x \in V$ and $L_{A^{-1}}L_A(x) = x, \forall x \in V$.

We have $L_AL_{A^{-1}}(x) = AA^{-1}x = x$ and $L_{A^{-1}}L_A(x) = A^{-1}Ax = x$.

**Defn 11.** Let $V$ and $W$ be vector spaces. We say that $V$ is isomorphic to $W$ if there exists a linear transformation $T : V \to W$ that is invertible. Such a linear transformation is called an isomorphism from $V$ onto $W$.

**Theorem 2.19.** Let $V$ and $W$ be finite-dimensional vector spaces (over the same field). Then $V$ is isomorphic to $W$ if and only if $\dim(V) = \dim(W)$.

**Proof.** ($\Rightarrow$) Let $T : V \to W$ be an isomorphism. Then $T$ is invertible and $\dim V = \dim W$ by the Lemma.

($\Leftarrow$) Assume $\dim V = \dim W$. Let $\beta = \{v_1, \ldots, v_k\}$ and $\gamma = \{w_1, w_2, \ldots, w_k\}$ be bases for $V$ and $W$, respectively. Define $T : V \to W$ where $v_i \mapsto w_i$, for all $i$.

Then $[T]_\beta = I$ which is invertible and Theorem 2.18 implies that $T$ is invertible and thus, an isomorphism.

**Fact 7.** Let $P \in M_n(F)$ be invertible. $W$ is a subspace of $F^n$ implies $L_P(W)$ is a subspace of $F^n$ and $\dim(L_P(W)) = \dim(W)$.

**Proof.** Let $x, y \in L_P(W)$, $a \in F$, then there exist $x'$ and $y'$ such that $Px' = x$ and $Py' = y$.

So $P(ax' + y') = aPx' + Py' = ax + y$. So $ax + y \in L_P(W)$. Also, we know $0 \in W$ and $L_P(0) = 0$ since $L_P$ is linear, so we have that $0 \in L_P(W)$. Therefore, $L_P(W)$ is a subspace.

Let $\{x_1, x_2, \ldots, x_k\}$ be a basis of $W$.

$$a_1P(x_1) + a_2P(x_2) + \cdots + a_kP(x_k) = 0$$

$\Rightarrow$ $P(a_1x_1 + a_2x_2 + \cdots + a_kx_k) = 0$

$\Rightarrow$ $a_1x_1 + a_2x_2 + \cdots + a_kx_k = 0$

$\Rightarrow$ $a_1 = a_2 = \cdots = a_k = 0$

$\Rightarrow$ $\{P(x_1), P(x_2), \ldots, P(x_k)\}$ is linearly independent.
To show that \( \{P(x_1), P(x_2), \ldots, P(x_k)\} \) spans \( L_P(W) \), we let \( z \in L_P(W) \). Then there is some \( w \in W \), such that \( L_P(W) = z \). If \( w = a_1x_1 + a_2x_2 + \cdots + a_kx_k \), we have

\[
z = Pw = P(a_1x_1 + a_2x_2 + \cdots + a_kx_k) = a_1P(x_1) + a_2P(x_2) + \cdots + a_kP(x_k).
\]

And with have that \( \{P(x_1), P(x_2), \ldots, P(x_k)\} \) is a basis of \( L_P(W) \) and \( \dim(W) = \dim(L_P(W)) \).

So far, we did not define the rank of a matrix. We have:

**Defn 12.** Let \( A \in M_{m \times n}(F) \). Then, \( \text{Rank}(A) = \text{Rank}(L_A) \). And the range of a matrix, \( R(A) \) is the same as \( R(L_A) \).

**Fact 8.** Let \( S \in M_n(F) \). If \( S \) is invertible, then \( R(S) = F^n \).

**Proof.** \( LS : F^n \rightarrow F^n \). We know \( S \) is invertible implies \( LS \) is invertible. We already know that, since \( LS \) is onto, \( R(LS) = F^n \). By the definition of \( R(S) \), we have \( R(S) = F^n \).

**Fact 9.** \( S, T \in M_n(F) \), \( S \) invertible and \( T \) invertible imply \( ST \) is invertible and its inverse is \( T^{-1}S^{-1} \).

**Proof.** This is the oldest proof in the book.

**Theorem 2.20.** Let \( V \) and \( W \) be finite-dimensional vector spaces over \( F \) of dimensions \( n \) and \( m \), respectively, and let \( \beta \) and \( \gamma \) be ordered bases for \( V \) and \( W \), respectively. Then the function

\[
\Phi : \mathcal{L}(V, W) \rightarrow M_{m \times n}(F)
\]

defined by \( \Phi(T) = [T]_\beta^\gamma \) for \( T \in \mathcal{L}(V, W) \), is an isomorphism.

**Proof.** To show \( \Phi \) is an isomorphism, we must show that it is invertible. Let \( \beta = \{v_1, v_2, \ldots, v_n\} \) and \( \gamma = \{w_1, w_2, \ldots, w_m\} \).

We define \( U : M_{m \times n}(F) \rightarrow \mathcal{L}(V, W) \) as follows.

Let \( A \in M_{m \times n}(F) \) and have \( i^{th} \) column:

\[
\begin{pmatrix}
a_{1,i} \\
a_{2,i} \\
\vdots \\
a_{n,i}
\end{pmatrix}
\]

So \( U \) maps matrices to transformations. \( U(A) \) is a transformation in \( \mathcal{L}(V, W) \). We describe the action of the linear transformation \( U(A) \) on \( v_i \) and realize that this uniquely defines a linear transformation in \( \mathcal{L}(V, W) \).

\[
\forall i \in [n], U(A)(v_i) = a_{1,i}w_1 + a_{2,i}w_2 + \cdots + a_{n,i}w_n.
\]

Then \( A = [U(A)]_\beta^\gamma \). Thus, \( \Phi U(A) = A \).

To verify that \( U(\Phi(T)) = T \), we see what the action is on \( v_i \).

\[
U(\Phi(T))(v_i) = U([T]_\beta^\gamma)(v_i) = t_{1,i}w_1 + t_{2,i}w_2 + \cdots + t_{n,i}w_n = T(v_i)
\]

**Cor 1.** Let \( V \) and \( W \) be finite-dimensional vector spaces over \( F \) of dimensions \( n \) and \( m \), respectively. Then \( \mathcal{L}(V, W) \) is finite-dimensional of dimension \( mn \).
Proof. This by Theorem 2.19. ■

**Theorem 2.22.** Let $\beta$ and $\gamma$ be two ordered bases for a finite-dimensional vector space $V$, and let $Q = [I_V]_\beta^\gamma$. Then

(a) $Q$ is invertible.
(b) For any $v \in V$, $[v]_\gamma = Q[v]_\beta$.

**Proof.** We see that

Let $\beta = \{v_1, \ldots, v_n\}$ and $\gamma = \{w_1, \ldots, w_n\}$ We claim the inverse of $Q$ is $[I_V]_\beta^\gamma$. We see that the $i^{th}$ column of $[I_V]_\beta^\gamma$ is $[I_V]_\gamma^\beta e_i$.

We have

\[
[I_V]_\beta^\gamma e_i = [I_V]_\gamma^\beta [I_V]_\gamma^\beta [w_i]_\gamma
= [I_V]_\gamma^\beta [I_V (w_i)]_\beta
= [I_V (I_V (w_i))]_\gamma
= [w_i]_\gamma
= e_i
\]

and

\[
[I_V]_\gamma^\beta [I_V]_\beta^\gamma e_i = [I_V]_\gamma^\beta [I_V]_\gamma^\beta [v_i]_\beta
= [I_V]_\gamma^\beta [I_V (v_i)]_\gamma
= [I_V (I_V (v_i))]_\beta
= [v_i]_\beta
= e_i
\]

We know, $[v]_\gamma = [I_V (v)]_\gamma = [I_V]_\beta^\gamma [v]_\beta = Q[v]_\beta$. ■

**Defn 13.** (p. 112) The matrix $Q = [I_V]_\beta^\gamma$ defined in Theorem 2.22 is called a change of coordinate matrix. Because of part (b) of the theorem, we say that $Q$ changes $\beta$-coordinates into $\gamma$-coordinates. Notice that $Q^{-1} = [I_V]_\gamma^\beta$.

**Defn 14.** (p. 112) A linear transformation $T : V \rightarrow V$ is called a linear operator.

**Theorem 2.23.** Let $T$ be a linear operator on a finite-dimensional vector space $V$, and let $\beta$ and $\gamma$ be ordered bases for $V$. Suppose that $Q$ is the change of coordinate matrix that changes $\beta$-coordinates into $\gamma$-coordinates. Then

\[
[T]_\beta = Q^{-1}[T]_\gamma Q.
\]

**Proof.** The statement is short for:

\[
[T]_\beta = Q^{-1}[T]_\gamma Q.
\]
Let $\beta = \{v_1, v_2, \ldots, v_n\}$. As usual, we look at the $i^{th}$ column of each side. We have
\[
Q^{-1}[T]_\gamma^\beta Q e_i = [I_V]_\gamma^\beta [T]_\gamma^\beta [I_V]_\gamma^\beta [v_i]_\beta = [I_V]_\gamma^\beta [T]_\gamma^\beta [I_V(v_i)]_\gamma = [I_V]_\gamma^\beta [T]_\gamma^\beta [v_i]_\gamma = [I_V](T(v_i))_\beta = T(v_i) = [T]_\beta^\gamma [v_i]_\beta = [T]_\beta^\gamma e_i
\]
The $i^{th}$ column of $[T]_\beta^\gamma$.

**Cor 1.** Let $A \in M_{n \times n}(F)$, and let $\gamma$ be an ordered basis for $F^n$. Then $[L_A]_\gamma = Q^{-1}AQ$, where $Q$ is the $n \times n$ matrix whose $j^{th}$ column is the $j^{th}$ vector of $\gamma$.

**Proof.** Let $\beta$ be the standard ordered basis for $F^n$. Then $A = [L_A]_\beta$. So by the theorem, $[L_A]_\gamma = Q^{-1}[L_A]_\beta Q$.

**Defn 15.** (p. 115) Let $A$ and $B$ be in $M_n(F)$. We say that $B$ is similar to $A$ if there exists an invertible matrix $Q$ such that $B = Q^{-1}AQ$.

**Defn 16.** (p. 119) For a vector space $V$ over $F$, we define the dual space of $V$ to be the vector space $L(V, F)$, denoted by $V^*$. Let $\beta = \{x_1, x_2, \ldots, x_n\}$ be an ordered basis for $V$. For each $i \in [n]$, we define $f_i(x) = a_i$ where $a_i$ is the $i^{th}$ coordinate of $[x]_\beta$. Then $f_i$ is in $V^*$ called the $i^{th}$ coordinate function with respect to the basis $\beta$.

**Theorem 2.24.** Suppose that $V$ is a finite-dimensional vector space with the ordered basis $\beta = \{x_1, x_2, \ldots, x_n\}$. Let $f_i(1 \leq i \leq n)$ be the $i^{th}$ coordinate function with respect to $\beta$ and let $\beta^* = \{f_1, f_2, \ldots, f_n\}$. Then $\beta^*$ is an ordered basis for $V^*$, and, for any $f \in V^*$, we have
\[
f = \sum_{i=1}^n f(x_i) f_i.
\]

**Proof.** Let $f \in V^\beta$. Since dim $V^* = n$, we need only show that
\[
f = \sum_{i=1}^n f(x_i) f_i,
\]
from which it follows that $\beta^*$ generates $V^*$, and hence is a basis by a Corollary to the Replacement Theorem.

Let
\[
g = \sum_{i=1}^n f(x_i) f_i.
\]
For $1 \leq j \leq n$, we have
\[
g(x_j) = \left(\sum_{i=1}^n f(x_i) f_i\right)(x_j) = \sum_{i=1}^n f(x_i) f_i(x_j) = \sum_{i=1}^n f(x_i) \delta_{i,j} = f(x_j).
\]
Therefore, \( f = g \).