1. Graphs Model Real World Problems

Exercise 1.1. Each of the following can be modelled by a graph. Explain what the vertices would represent and what the edges would correspond to.

(1) A road map
(2) A molecule
(3) A subway system
(4) A family tree
(5) Jobs and applicants for those jobs.
Exercise 2.1.

Draw

(1) a simple graph
(2) a non-simple graph with no loops
(3) a non-simple graph with no multiple edges,
    each with five vertices and eight edges.

Exercise 2.2. Draw a graph with degree sequence \((3, 3, 5, 5, 5, 5)\). Does there exist a simple graph with this degree sequence?

Exercise 2.3. Draw a graph with degree sequence \((2, 3, 3, 4, 5, 5)\). Does there exist a simple graph with this degree sequence?

Exercise 2.4. Write down the adjacency matrix of the following graph.

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Exercise 2.5. Draw the graph whose adjacency matrix is given here:

\[
A = \begin{bmatrix}
0 & 1 & 1 & 2 & 0 \\
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 \\
2 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
\end{bmatrix}
\]

Exercise 2.6. Give an example of each of the following:

(1) a bipartite graph that is regular of degree 5.
(2) a bipartite platonic graph
(3) a complete graph that is a wheel
(4) a cubic graph with 11 vertices
(5) a graph (other than \(K_5\), \(K_{4,4}\), or \(Q_4\)) that is regular of degree 4.
Exercise 2.7. If $D$ is a digraph of order $n$ and size $m$ with $V(D) = \{v_1, v_2, \ldots, v_n\}$ then

$$
\sum_{i=1}^{n} od(v_i) = \sum_{i=1}^{n} id(v_i) = m.
$$
3. Subgraphs and Isomorphisms

Assume all graphs are simple

**Exercise 3.1.** A simple graph that is isomorphic to its complement is **self-complementary**.

1. Prove that if $G$ is self-complementary, then $G$ has $4k$ or $4k + 1$ vertices where $k$ is an integer.
2. Find all self-complementary graphs with 4 and 5 vertices.
3. Find a self-complementary graph with 8 vertices.

**Exercise 3.2.** Prove that either a graph or its complement is connected.

**Exercise 3.3.** Show that every simple graph contains two vertices of equal degree.

**Exercise 3.4.** Suppose there are $n$ married couples at a party, husband and wife. The host and hostess are also husband and wife, they count as one of the $n$ couples. Various people shake hands, yet no husband shakes hands with his wife.

The $2n - 1$ people other than the host each say they shook hands with different numbers of people. That is, no two shook hands with the same number of people. With how many people did the hostess shake hands? Prove your claim by induction on $n$.

**Exercise 3.5.** Draw 2 non-isomorphic 3 regular graphs on 6 vertices.

**Exercise 3.6.** Consider the graph $G$ below. Draw all subgraphs of $G$ that use the vertex set \{a, b, e, f\}.

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**Exercise 3.7.** Describe all 2-regular graphs.
**Exercise 3.8.** Determine which of the graphs, \( \{H, J, K\} \) are subgraphs of the graph \( G \). Which are induced subgraphs. Need pic.

**Exercise 3.9.** Show that the following 2 graphs are not isomorphic. Need pic.

**Exercise 3.10.** Show that the following 2 graphs are isomorphic. Need pic.

**Exercise 3.11.** Show that the sequence \( (n-1, 3, 3, 3, \ldots, 3) \) of length \( n \geq 4 \) is graphical. Try \( n = 4 \), then try \( n = 5 \), and so on.

**Exercise 3.12.** Prove that \( C_4 \) and \( K_{2,2} \) are isomorphic.

**Exercise 3.13.** Draw two different 4-regular graphs on 7 vertices and prove that they are non-isomorphic.

**Exercise 3.14.** Show that every connected graph \( G \) of order at least two contains vertices \( x \) and \( y \) such that both \( G - x \) and \( G - y \) are connected.

**Exercise 3.15.** (1) Show that every graph \( G \) has a bipartition \( V(G) = U \cup W \) such that \( e(U, W) \geq \frac{1}{2} e(G) \), where \( e(U, W) \) is the number of edges with one vertex in \( U \) and one vertex in \( W \) and \( e(G) \) is the number of edges in \( G \).

(2) Show also that if \( G \) is cubic of order \( n \), then we may demand that \( e(U, W) \geq n \).

**Exercise 3.16.** Show that every graph with average degree \( d \) contains a subgraph of minimal degree at least \( d/2 \).
Exercise 3.17. Show that up to isomorphism, there is a unique connected graph with degree sequence \((2, 2, \ldots, 2, 1, 1)\).

Exercise 3.18. Prove the set definition of \(Q_k\) is equivalent to the \(\{0, 1\}\)-sequence definition of \(Q_k\).

Exercise 3.19. (-) For every graph \(G\), \(\text{Aut}(G) \cong \text{Aut}(\overline{G})\).

Exercise 3.20. (-) The order \(|\text{Aut}(G)|\) of the automorphism group of a graph \(G\) of order \(n\) divides \(n!\), and \(|\text{Aut}(G)| = n!\) if and only if \(G = K_n\) or \(G = \emptyset_n\).

Exercise 3.21. (-) The number of distinct labellings of a graph \(G\) of order \(n\) from a set of \(n\) labels is

\[
\frac{n!}{|\text{Aut}(G)|}.
\]

Exercise 3.22. Every edge transitive graph without isolated vertices is either vertex-transitive or bipartite.

Exercise 3.23. If \(G\) is an \((n, m)\) graph with \(n \geq 3\) then \(n, m\) and all degrees of \(G\) are determined from the \(n\) subgraphs \(G - v, v \in V(G)\).

Exercise 3.24. (-) Every regular graph is reconstructible.

Exercise 3.25. (-) Connectedness is recognizable.

Exercise 3.26. (+)Disconnected graphs of order at least 3 are reconstructible.
4. Paths and Cycles

Exercise 4.1. For which values of $n$ is $K_n$ Eulerian?

Exercise 4.2. Which complete bipartite graphs are Eulerian?

Exercise 4.3. Which Platonic graphs are Eulerian?

Exercise 4.4. For which values of $n$ is the wheel $W_n$ Eulerian?

Exercise 4.5. Is there a knight’s tour of an 8x8 checkerboard?

Exercise 4.6. Which of the following graphs are Hamiltonian?

(1) the complete graph $K_5$
(2) the complete bipartite graph $K_{2,3}$
(3) the graph of the octahedron
(4) the wheel $W_6$
(5) the 4-cube, $Q_4$.

Exercise 4.7. Prove that the Petersen graph is non-Hamiltonian.

Exercise 4.8. Prove that the radius and diameter of a graph satisfy the inequalities

$$\text{rad}G \leq \text{diam}G \leq 2\text{rad}G$$

and both inequalities are best possible.

Exercise 4.9. If $A$ is the adjacency matrix of a graph $G$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$, then the $(i,j)$ entry of $A^k$, for $k \geq 1$ is the number of different $v_i, v_j$-walks of length $k$ in $G$.

Exercise 4.10. Let $G$ be a multi-graph. Prove that every odd length closed walk in $G$ contains an odd length cycle.
5. Trees

Exercise 5.1. Draw all non-isomorphic trees that have 5 or fewer vertices.

Exercise 5.2. What is the Complexity of the following decision problem? Given a graph $G$, is $G$ a tree?

Exercise 5.3. Show that $d_1 \leq d_2 \leq \cdots \leq d_n$ is the degree sequence of a tree $\iff d_1 \geq 1$ and $\sum_{i=1}^{n} d_i = 2n - 2$.

Exercise 5.4. Let $T$ be a tree of order $k$, and let $G$ be a graph with $\delta(G) \geq k - 1$. Then $T$ is a subgraph of $G$. 
6. **Planar Graphs**

**Exercise 6.1.** Prove that every simple planar graph has 2 vertices of degree at most 5.

**Exercise 6.2.** Let $G$ be a maximal simple plane graph. Prove that $G^*$, the dual of $G$, is 2-edge-connected and 3-regular.

**Exercise 6.3.** Find all possible dual relationships between the platonic graphs.

**Exercise 6.4.** Prove that every simple outer-planar graph has a vertex of degree 2.

7. **Independence and Coloring**

**Exercise 7.1.** If $G$ is a connected 3-critical graph then $G$ is an odd-cycle.

**Exercise 7.2.** If $G$ is an $n$ critical graph with a 2-vertex cut set $\{u, v\}$. Then

$$\deg u + \deg v \geq 3n - 5.$$

8. **Matchings**

**Exercise 8.1.** If $G$ is a bipartite graph with $\Delta(G) = k$, show that $G$ is the subgraph of a $k$-regular bipartite graph.