Application - Hospitals
Queens on board - puzzle
Minimal dominating set. $S \subseteq V(G)$, every vertex not in $S$ has a neighbor in $S$.
Minimum dominating set, $\gamma(G)$
Independent set:
Independence number, $\alpha(G)$
Theorem: $\gamma(G) \leq \alpha(G)$
Vertex Cover. $Q \subseteq V(G)$, contains at least one endpoint of every edge.
Vertex covering number, $\beta(G)$.
Size of a maximum matching $\alpha'(G)$. 
Theorem 28. $S \subseteq V(G)$ is an independent set if and only if $\tilde{S}$ is a vertex cover. Hence, $\alpha(G) + \beta(G) = |V(G)|$.

Proof. Let $S$ be an independent set. Then every edge is incident to at least one vertex of $(S)$. Conversely, if $\tilde{S}$ covers all edges, there are not edges among pairs of vertices in $S$. Hence every maximum independent set is the complement of a minimum vertex cover. □

Theorem 29. (Konig; Egevary) If $G$ is a bipartite graph, then $\alpha'(G) = \beta(G)$.

Proof. Let $V(G) = X \cup Y$ where $X$ and $Y$ are the partite sets. Let $Q$ be a vertex cover and $M$ any matching. It must be that $|Q| \geq |M|$, since distinct vertices are needed to cover the edges of $M$. Hence we can conclude that $\alpha'(G) \geq \beta(G)$.

Let $Q$ be a smallest vertex cover. We construct a matching of size $|Q|$, thus proving that $\beta(G) \geq \alpha'(G)$.

Let $R = Q \cap X$ and $T = Q \cap Y$. Let $H$ be the subgraph of $G$ induced on $R \cup (Y - T)$ and $H'$ be the subgraph of $G$ induced on $T \cup (X - R)$.

We will use Hall’s Theorem to show that $H$ has a matching that saturates $R$ and $H'$ has a matching that saturates $T$. As $H$ and $H'$ are disjoint, the two matchings together give a matching with $|Q|$ edges.

Since $R \cup T$ is a vertex cover, $G$ has no edge between $Y - T$ and $X - R$. For each $S \subseteq R$, we consider $N_H(S)$. If $|N_H(S)| < |S|$, we could substitute $N_H(S)$ for $S$ in $Q$ (since $N_H(S)$ covers all edges incident to $S$ that are not covered by $T$) and have a smaller vertex cover of $G$. But that contradicts that $Q$ is a smallest vertex cover of $G$. Thus Hall’s condition is satisfied in $H$ and there is a matching with $|R|$ edges.

Similarly there is a matching with $|T|$ edges in $H'$. □