Lesson 8 - Matchings

Color classes of edges in a proper edge coloring form matchings.

Matchings

Perfect matchings

Thm: \( \chi(K_n) = n \) if \( n \) is odd. \( \chi(K_n) = n - 1 \) if \( n \) is even.

Matchings in bipartite graphs.

Hall’s condition.

Hall’s marriage theorem

Applications of Hall’s Theorem

System of distinct representatives

Applications: In bipartite, job and applicants

Applications: In general graph: class trips, vertices = kids, edges=friends, pair up so friends sit together on bus
Theorem 24. \( \chi(K_n) = n \) if \( n \) is odd. \( \chi(K_n) = n - 1 \) if \( n \) is even.

Definition 5. A matching in an undirected graph \( G \) is a set of pairwise disjoint edges. The vertices belonging to the edges in the matching are called \textbf{saturated}. Others are \textbf{unsaturated}. A \textbf{perfect matching} or \textbf{complete matching} is a matching that saturates every vertex of \( G \). A \textbf{maximum matching} is one that has the most edges of all matchings in \( G \). A \textbf{maximal matching} may not be a maximum matching, but it cannot be extended.

Definition 6. Given a matching \( M \), an \textbf{M-alternating path} is a path that alternates between edges in \( M \) and edges not in \( M \). An \textbf{M-alternating path} \( P \) that begins and ends with \( M \)-unsaturated vertices is an \textbf{M-augmenting path}.

Proposition 1. Replacing \( M \cap E(P) \) by \( E(P) - M \) produces a new matching \( M' \) with one more edge than \( M \).

Example 1. See Figure 1. Let \( M \) be the matching \( \{1C, 3D, 4E\} \) and \( P \) the \( M \)-augmenting path \( P : A, 1, C, 4, E, 5 \). Then \( M \cap E(P) = \{1C, 4E\} \) and \( E(P) - M = \{1A, 4C, 5E\} \). We get \( M' = (M - E(P)) \cup (E(P) - M) = \{3D, 1A, 4C, 5E\} \).

Definition 7. If \( G \) and \( H \) are two graphs on the same vertex set \( V \), then the \textbf{symmetric difference} \( G \triangle H \) is the
graph with vertex set $V$ whose edges are all edges appearing in exactly one of $G$ and $H$. This is also used for sets of edges: If $M$ and $M'$ are 2 matchings then $M \triangle M' = (M \cup M') - (M \cap M')$.

**Lemma 24.1.** A matching $M$ in a graph $G$ is maximum if and only if $G$ has no $M$-augmenting path.

**Proof:** ($\Rightarrow$) Suppose $G$ has an $M$-augmenting path, $P$. Then $M' = (M - E(P)) \cup (E(P) - M)$ is a larger matching than $M$.

($\Leftarrow$) Suppose there is a matching $M'$ in $G$ larger than $M$. Let $F$ be the subgraph of $G$ on $V(G)$ with edges $E(F) = M \triangle M'$. Clearly, $\Delta(F) \leq 2$. So $F$ consists of disjoint paths and cycles. Furthermore, every path or cycle in $F$ alternates between edges in $M$ and $M'$. So each cycle in $F$ must be even length. But $|M'| > |M|$. So there is a component which is a path that begins and ends with an edge of $M'$. This is an $M$-augmenting path. $\square$

**Theorem 25.** P. Hall. Let $G$ be a bipartite graph with partite sets $A$ and $B$, $|A| \leq |B|$. Then $\forall X \subseteq A, |N(X)| \geq |X|$ if and only if there exits a matching of $A$ into $B$. ($A$ is saturated.)

**Proof:** $\Leftarrow$ The necessity is clear.

$\Rightarrow$ For the sufficiency, we suppose $|N(S)| \geq |S|$ for all $S \subseteq A$ and consider a maximum matching $M$. By the Lemma 24.1, there is no $M$-augmenting path. Suppose $M$ does not saturate $A$. Let $u \in A$ be an unsaturated vertex. We know that $u$ has no neighbors that are not incident to edges of $M$ or else $M$ would not be maximal. Let $S \subseteq A$ be vertices reachable by an $M$-alternating path starting at $u$ and $T \subseteq B$ be vertices reachable from $u$ by an $M$-alternating path. All the paths end in $S$ since there are no $M$-augmenting
paths. Thus, if \( s \in S - u \), it has no neighbors outside of \( T \), so \( N(S - u) \subseteq T \). But since every vertex of \( T \) has a matching edge in \( S - u \), we see that \( |T| \leq |S - u| \).

We see that for \( X = S \), \( |N(S)| \leq |T| \leq |S - u| \), so \( |S| = |S - u| + 1 > |N(S)| \), which is a contradiction. \( \square \)

**Exercise 2.** If \( G \) is a bipartite graph with \( \Delta(G) = k \), show that \( G \) is the subgraph of a \( k \)-regular bipartite graph.

**Theorem 26.** If \( G \) is a \( k \)-regular bipartite graph then it is in class 1.

**Proof:** In this case to prove that \( G \) is in class 1 is to show that the edges are \( k \) colorable.

Suppose \( G \) has partite sets \( A \) and \( B \). Since \( G \) is \( k \)-regular, we know that \( |A| = |B| \).

The proof is by induction on \( k \).

If \( k = 1 \), there is a perfect matching and it contains all the edges of \( G \). Thus the edges can be colored with 1 color.

We will show that Hall’s condition is satisfied. Let \( X \subseteq A \). We notice that \( E(X, N(X)) \subseteq E(N(X), A) \). So,

\[
 k|N(X)| = |E(N(X), A)| \\
 \geq |E(X, N(X))| \\
 = k|X| \\
 |N(X)| \geq |X|
\]

Thus \( G \) has a perfect matching. We consider these edges to be colored with color \( c_k \).

We remove these edges, thus obtaining a \( k - 1 \)-regular bipartite graph. By induction, the edges are colorable with \( k - 1 \) colors. Thus obtaining a \( k \) coloring of \( G \).

**Theorem 27.** If \( G \) is bipartite then it is in class 1.