Lesson 6 - Planar Graphs

Planar graph
plane drawing - plane graph
face
length of a face
infinite face

Thm 12: Euler. Let $G$ be a plane drawing of a connected planar graph. With $n$, $m$, $f$, denote respectively the number of vertices, edges and faces. Then $n - m + f = 2$.
Cor 12.1: $k$ components $n - m + f = k + 1$
Cor 12.2: $m \leq 3n - 6$
Cor 12.3: $m \leq 2n - 4$ (triangle free)

Thm 13: $K_{3,3}$ and $K_5$ are non-planar.

Thm 14: Any graph containing a non-planar subgraph is non-planar.

homeomorphic - subdivison,

Thm 15: Any graph homeomorphic to a non-planar graph is non-planar.

Thm 16: Any graph homeomorphic to either $K_{3,3}$ or $K_5$ is non-planar.

Thm 17: Any graph containing a subgraph homeomorphic to either $K_{3,3}$ or $K_5$ is non-planar.

Thm 18: (Kuratowski) A graph is planar if and only if it contains no subgraph homeomorphic to $K_{3,3}$ or $K_5$.

Thm 19: Every planar graph contains a vertex of degree at most 5

Thm 20: 5 - color theorem
Crossing number $cr(G)$
Printed circuits
Thickness $t(G)$
Complexity question: Decide if $G$ is planar.
**Theorem 12. Euler.** Let $G$ be a plane drawing of a connected planar graph. With $n$, $m$, $f$, denote respectively the number of vertices, edges and faces. Then $n - m + f = 2$.

**Proof:** The proof is by induction on the number of edges. The base case is when there are no edges. Then since $G$ is connected, it is the trivial graph with $n = 1$, $m = 0$, and $f = 1$. We see that $n - m + f = 1 - 0 + 1 = 2$. Let $m > 1$. We assume that a connected plane graph that has $m'$ edges with $m > m' \geq 1$, $n'$ vertices and $f'$ faces then $n' - m' + f' = 2$.

Now take a connected plane graph $G$ on $m$ edges, $n$ vertices, and $f$ faces. If $G$ is a tree then we know that $m = n - 1$ and $f = 1$. In this case we have: $n - m + f = n - (n - 1) + 1 = 1 + 1 = 2$. If $G$ is not a tree, not every edge is a cut edge. If $uv$ is not a cut edge then $uv$ has is bordered by two different faces, and $G - uv$ is connected with 1 less face than $G$. Then $G - uv$ satisfies the induction hypothesis and we have that $n - (m - 1) + f - 1 = 2$ which implies that $n - m + f = 2$. □

**Corollary 12.1.** Let $G$ be a plane drawing of a planar graph that has $k$ components, $n$ vertices, $m$ edges, and $f$ faces. Then $n - m + f = k + 1$.

**Proof:** Suppose $G$ has $k$ components. For each $i \in [k]$ suppose component $C_i$ has $n_i$ vertices, $m_i$ edges, and $f_i$ faces. Each one is connected, and so by Euler’s Formula. For all $i \in [k]$, $n_i - m_i + f_i = 2$. Adding the $k$ equations we get: $n_1 + \cdots + n_k - (m_1 + \cdots + m_k) + (f_1 + \cdots + f_k) = 2k$. Substituting $n_1 + \cdots + n_k = n$, $m_1 + \cdots + m_k = m$ and $f_1 + \cdots + f_k = f + k - 1$, gives $n - m + f + k - 1 = 2k$. Which of course implies, $n - m + f = 2k - k + 1 = k + 1$. □

**Definition 2.** The length of a face in a plane graph $G$ is the total length of the closed walk(s) in $G$ bounding the face.
Note that a cut edge is bordered on both sides by the same face and contributes 2 times to the length of the face.

Corollary 12.2. If $G$ is a planar graph with $n$ vertices and $m$ edges then $m \leq 3n - 6$. Moreover, if $G$ is maximal planar then, $m = 3n - 6$.

**Proof:** Embed $G$ in the plane. First assume that $G$ is maximal planar. Then $G$ is connected and every face must be bordered by 3 edges. That is the length of each face, $F$, $\ell(F)$ is 3. We have that $\sum_{i=1}^{f} \ell(F_i) = 2m$ and on the other hand, $\sum_{i=1}^{f} \ell(F_i) = 3f$. By Euler’s Formula we may substitute $f = m - n + 2$. Thus, $2m = 3(m - n + 2)$, so that $m = 3n - 6$. If $G$ is not maximal planar, we can add edges to $G$ to obtain $G'$ which is maximal planar and has $m'$ edges. Then $m < m' = 3n - 6$. $\square$

**Corollary 12.3.** If $G$ is a planar graph with $n$ vertices, $m$ edges, and no 3-cycles then $m \leq 2n - 4$.

**Proof:** The argument is similar to that of the last corollary. Except, in this case, we have that the length of each face is at least 4. Thus obtaining $4f \leq 2m$ and after substituting $f = m - n + 2$, we get $4(m - n + 2) \leq 2m$. So that $m \leq 2n - 4$. $\square$

**Theorem 13.** $K_{3,3}$ and $K_5$ are non-planar.

**Proof:** Suppose $K_{3,3}$ is planar. By Corollary 12.3, for $n = 6$ and $m = 9$, $9 \leq 2(6) - 4 = 8$. But this is a contradiction, so $K_{3,3}$ cannot be planar.

Suppose $K_5$ is planar. By Corollary 12.2, for $n = 5$ and $m = 10$, $10 \leq 3(5) - 6 = 9$. But this is a contradiction, so $K_5$ cannot be planar. $\square$

**Theorem 14.** Any graph containing a non-planar subgraph is non-planar.
Proof: The proof should be clear.

Theorem 15. Any graph homeomorphic to a non-planar graph is non-planar.

Proof: The proof should be clear.

Theorem 16. Any graph homeomorphic to either $K_{3,3}$ or $K_5$ is non-planar.

Proof: This is due to Theorem 13 and Theorem 15.

Theorem 17. Any graph containing a subgraph homeomorphic to either $K_{3,3}$ or $K_5$ is non-planar.

Proof: This is due to Theorem 16 and Theorem 14.

Theorem 18. Kuratowski. A graph is planar if and only if it contains no subgraph homeomorphic to $K_{3,3}$ or $K_5$.

Theorem 17 gives the sufficiency, ($\Rightarrow$). For the necessity, ($\Leftarrow$), we must first prove a number of lemmas.

Definition 3. A graph is 2-connected if there are no cut vertices. A subgraph of a graph $G$ is called a block if it is a maximal 2-connected subgraph. If a graph is 2-connected then it might be called a block.

The meaning of “maximal” in the above definition is this. If $B$ is a block of $G$ and $H$ is a 2-connected subgraph of $G$ containing $B$, then $B = H$. A block of $G$ may contain vertices which are cut-vertices of $G$ but will not be cut vertices in the block. A graph $G$ with at least one cut vertex has more than one block. We notice that the blocks form a sort of tree structure and so there are blocks which behave like leaves in a tree and there will be at least 2 of them.

The next lemma says something about cycles in blocks.
Lemma 18.1. A graph $G$ of order $p \geq 3$ is a block if and only if every two vertices of $G$ lie on a common cycle of $G$.

Proof: Suppose every 2 vertices of $G$ lie on a common cycle. Then, no vertex could be a cut vertex. For, suppose $v$ is a cut vertex, then $\exists u, w$ such that $v$ is on every $u, w$-path. But then there is no cycle containing both $u$ and $w$. Since $G$ has no cut vertex, it is a block.

Now suppose $G$ is a block and for the sake of contradiction, suppose for vertex $u$ there is some vertex, not on a cycle with $u$. Let $U$ be the set containing $u$ and all vertices that are on common cycles with $u$. Since $G - u$ is still connected, every pair of vertices in $N(u)$ are connected by a path in $G - u$. Therefore, $N(u) \subseteq U$.

We are assuming $V - U$ is nonempty. Let $v \in V - U$. Consider a $u, v$-path $W : u = u_0, u_1, \ldots u_n = v$. Let $i$ be the smallest integer $2 \leq i \leq n$ such that $u_i \notin U$. Then there exists a cycle $C$ containing $u$ and $u_{i-1}$.

Because $u_{i-1}$ is not a cut vertex, there exists a $u_i, u$-path $P : u_i = v_0, v_1, v_2, \ldots v_m = u$ not containing $u_{i-1}$. The only vertex common to both $P$ and $C$ cannot be $u$ because if so then following $C$ from $u$ to $u_{i-1}$ then along edge $u_{i-1}u_i$ to $u_i$ followed by $P$ back to $u$ is a cycle containing $u$.
Let $j$ be the smallest integer $1 \leq j \leq m - 1$ such that $v_j$ is on $C$. Now we have a cycle containing both $u$ and $u_i$, contradicting that $V - U$ is nonempty. Here is the cycle. Follow $P$ from $u_i$ to $v_j$, $C$ toward $u$, through $u$ back to $u_{i-1}$, then use the edge $u_{i-1}u_i$ back to $u_i$. □

**Lemma 18.2.** Let $G$ be a connected graph with one or more cut-vertices. Then among the blocks of $G$, there are at least two, each of which has exactly one cut vertex of $G$.

**Proof:** Form the following bipartite graph $H$ with bi-partitions $A$ and $B$. In $A$, include all the cut vertices of $G$ and in $B$, include a vertex $B_i$ for each block $B_i$ of $G$. We include an edge $b_i$ to $B_j$ in $H$ whenever $b_i \in B_j$. 

**Claim:** $H$ is a tree.

Suppose there is a cycle in $H$. Then it is of the form, $a_1, B_1, a_2, B_2, \ldots, B_k, a_1$ and corresponds to a cycle in $G$ within blocks $B_1, \ldots, B_k$. But then the previous theorem, $B_1 \cup B_1 \cup \cdots \cup B_k$ is a block of $G$. So for each $i$, $B_i$ is not maximally 2-connected in $G$. Thus, $H$ has no cycles. Clearly, $H$ is connected since $G$ is connected. Since $H$ is a tree, there are at least 2 end vertices, which are blocks in $G$ that contain exactly one cut vertex. □

**Definition 4.** A block containing one cut vertex is called an end-block.

Before we state the next lemma, we give a definition.

**Definition 5.** A block $G$ is called a critical block if $\forall v$, $G - v$ is not a block.

**Lemma 18.3.** If $G$ is a critical block of order $p \geq 4$, then $G$ contains a vertex of degree 2.

**Proof:** Clearly no vertex in a block $G$ is of degree 1. For each vertex $x$ of $G$, there exists another vertex $y$ of $G - x$ such
that $G - x - y$ is disconnected. Note that each component of $G - x - y$ must contain some vertices adjacent to $x$ and some adjacent to $y$. Otherwise we would have a cut vertex.

Among all such pairs $x, y$ of vertices of $G$, let $u, v$ be a pair such that $G - u - v$ contains a component $G_1$ of minimum order $n$. If $n = 1$ then that vertex in $G_1$ is of degree 2, since it must be adjacent to both $u$ and $v$.

Assume $n \geq 2$. Let $H = \langle V(G_1) \cup \{u, v\} \rangle$, and $G_2 = G - V(H)$. Let $w_1 \in V(G_1)$. Let $w_2 \in G - w_1$ be such that $G - w_1 - w_2$ is disconnected. Consider the following cases.

**Case 1. Suppose $w_2 \in V(H)$.**

See Figure 2. Since we know that no vertices of $G_2$ are adjacent to $w_1$ and $< V(G_2) \cup \{u\} >$ and $< V(G_2) \cup \{v\} >$ are connected, some component of $G - w_1 - w_2$ is smaller than $G_1$. So this case cannot happen.

**Case 2. Suppose $w_2 \in V(G_2)$.**

See Figure 3. Each component of $G - w_1 - w_2$ contains some vertices of $H$ and some vertices of $G_2$ due to the fact that each component must have vertices adjacent to each of $w_1$ and $w_2$. Since $G - u - v$ has the component $H - u - v$ and $G_2$ is the union of the remaining components, at least one of the vertices
Figure 3. $w_2 \in V(G_2)$

$u$ and $v$ are on every path between vertices in $H - u - v$ and $G_2$. Thus if $u$ and $v$ were in the same component of $G - w_1 - w_2$, there would be some component of $G - w_1 - w_2$ without vertices adjacent to $w_1$ or without vertices adjacent to $w_2$.

So, $H - w_1$ must contain at least 2 components, one containing $u$, call it $H_u$ and one containing $v$, call it $H_v$. If $H - w_1$ contains another component then again, that component would have no vertices adjacent to $w_2$.

If either $H_u$ or $H_v$ is trivial, then its only vertex would be of degree 2, adjacent to both $w_1$ and $w_2$. So assume $H_u$ and $H_v$ are nontrivial.

Now consider $G - w_1 - u$. This has a component contained in $H - w_1 - u$, namely $H_u - u$. We have $|H| = n + 2$, the sets $H_u, H_v, \{w_1\}$ form a partition of $H$, and $|H_v| \geq 2$. So $|H_u - u| \leq n - 1$. We again reach a contradiction to the way $u$ and $v$ were chosen. □

**Lemma 18.4.** A graph is planar if and only if each of its blocks is planar.

**Proof:** The graph $G$ is planar if and only if each of its components is planar, so assume $G$ is connected. Trivially, if $G$ is planar then each block is planar. It remains to show that if each
block is planar then \( G \) is planar. This is proved by induction on \( k \), the number of blocks.
Suppose \( k = 1 \). Then \( G \) is planar.
Now suppose \( G \) is a graph with \( k \geq 2 \) blocks and suppose for all graphs \( H \) with \( k - 1 \) planar blocks, \( H \) is planar.
Let \( B \) be an end-block of \( G \) with cut vertex \( v \). Then \( H = G - (B - v) \) has \( k - 1 \) planar blocks and so is planar by the induction assumption. Take any region \( R \) of the planar embedding of \( H \) which has \( v \) on its border. Consider a planar embedding of \( B \) with \( v \) on its exterior region. Now, \( B \) can be embedded in \( R \) while identifying \( v \) in \( B \) with \( v \) in \( H \). \( \square \)

**Lemma 18.5.** If a block contains no subdivision of \( K_5 \) or \( K_{3,3} \), then it is planar.

**Proof:** Suppose it’s not true. Of all non-planar blocks that contain no subdivision of \( K_5 \) or \( K_{3,3} \), pick \( H \) of minimum size, number of edges.

**Claim:** \( \delta(H) \geq 3 \).
If \( \delta(H) = 1 \) then \( H \) is not a block. Suppose \( v \in V(H) \) such that \( \deg v = 2 \) with \( vu, vw \in E(H) \). Then we would have one of the following two cases.

**Case 1.** \( uw \in E(H) \).
In this case, \( H - v \) is still a block and still has no subdivision of \( K_5 \) or \( K_{3,3} \). But \( H - v \) is smaller in size than \( H \) and so must be planar because of the way we chose \( H \). Clearly if \( H - v \) can be embedded in the plane then so can \( H \).

**Case 2.** \( uw \notin E(H) \).
Again, in this case, \( H - v + uw \) is still a block and still has no subdivision of \( K_5 \) or \( K_{3,3} \). For, suppose \( H_1 \) is a subdivision of
$K_5$ or $K_{3,3}$ in $H - v + uw$. Then $H$ would contain a subdivision of $H_1$. Thus again we see that there would be a planar embedding of $H$ in this case as well.

This proves the claim. Now we know by Theorem 18.3 that $H$ is not a critical block. Thus, there exists an edge $e = uv$ such that $H - e$ is still a block.

Set $H' = H - e$. $H'$ is planar since $H$ is of minimum size.

By Theorem 18.1, we know that $H'$ contains a cycle containing both $u$ and $v$. Among all planar embeddings of $H'$, choose one that has a cycle $C$ with the maximum number of interior regions. Let

$$C : u = v_0, v_1, \ldots, v_i = v, v_{i+1}, \ldots, v_n = u$$

where $1 < i < n - 1$.

**Observation 1:** $H$ is non-planar so both interior and exterior subgraphs are nonempty. Otherwise we can add the edge $uv$.

**Observation 2:** No 2 vertices are connected by a path embedded in the exterior of $C$ among

$$\{v_0, v_1, \ldots, v_i\}$$

or $$\{v_i, v_{i+1}, \ldots, v_n\}.$$ For, such a path cannot be embedded in the exterior region due to our choice of $C$ having the most number of interior regions.

By these two observations, we see that there must exist a $v_j - v_k$ path $P$ in the exterior with $0 < j < i < k < n$ that doesn’t intersect $C$ anywhere else or have any other paths connecting $P$ to $C$ at vertices other than $v_j$ or $v_k$. See Figure 4

Consider the subgraph $H' - (C - \{v_j, v_k\})$. Let $H_1$ be the component of this subgraph that contains the path $P$. By the choice of $C$, $H_1$ cannot be switched to the interior of $C$ in a
plane manner. This fact and the fact the edge $e = uv$ cannot be inserted in a plane manner imply that the interior of the subgraph $H'$ must contain one of the following:

1. A $v_r, v_s$-path, $Q$, $0 < r < j$, $i < s < k$, (or, equivalently, $j < r < i$ and $k < s < n$) none of whose vertices different from $v_r$ and $v_s$ belongs to $C$.

2. A vertex $w$ not on $C$ that is connected to $C$ by three internally disjoint paths such that the end-vertex of one such path $P'$ is one of $v_0, v_j, v_i$, and $v_k$. If $P'$ ends at $v_0$, the end-vertices of the other paths are $v_r$ and $v_s$ with $j \leq r < i$ and $i < s \leq k$ but not both $r = j$ and $s = k$ hold. If $P'$ ends at any of the other vertices, $v_j, v_i$ or $v_k$, there are three analogous cases.

3. A vertex $w$ not on $C$ that is connected to $C$ by three internally disjoint paths $P_1, P_2, P_3$ such that the end-vertices of the paths are three of the four vertices $v_0, v_j, v_i, v_k$. Say $v_k$ is the missing one, and $P_1$ connects $v_0$ to $w$ and $P_2$ connects $v_i$ to $w$. Then there is also a vertex $x \neq v_0, w, v_i$ on one of the paths $P_1$ or $P_2$ and a path $P_4$ from $x$ to...
$v_k$. The remaining choices of the missing vertex produce analogous results.

(4) A vertex $w$ not on $C$ that is connected to the vertices $v_0, v_j, v_i, v_k$ by four internally disjoint paths.

Figure 5 depicts the four situations given above. First, convince yourself that these are the only possibilities. Then observe that when we include the edge $e = uv$, in the first three cases, a subdivision of $K_{3,3}$ can be found and in the fourth case a subdivision of $K_5$ exists. This is of course a contradiction and completes the proof. □

**Proof of Theorem 18:** Given that $G$ contains no subgraph homeomorphic to either $K_5$ or $K_{3,3}$, we know that none of its
blocks contains one of these subgraphs. So all the blocks of $G$
are planar by Lemma 18.5. Then also $G$ is planar by Lemma
18.4. □

**Theorem 19.** Every planar graph has a vertex of degree at
most 5.

**Proof:** If we suppose not, then $6n < \sum_{i=1}^{n} \deg(v_i) = 2m$. But
this contradicts Corollary 12.2.

**Theorem 20.** Every planar graph is 5-colorable.

**Proof:** The proof is by induction on the order, $n$, of the graph.
The base case is when $n = 1$ and the theorem is trivially true
for any graph on 1 vertex.
Now suppose $n > 1$ and any planar graph on fewer than $n$
vertices can by 5-colored.
Suppose $G$ is a planar. If $G$ has a vertex $v$ of degree at most 4,
we can color $G - v$ by induction and then use a color not used
on $N(v)$ for $v$.
If not, then $G$ has a vertex, $v$, of degree 5. We color $G - v$ by
induction and look at the colors used on $N(v)$ for $v$.
For $\{i, j\} \subset [5]$, let $H(i, j)$ be the subgraph of $G$ induced on the
vertices that are colored with a color in $\{c_i, c_j\}$. If for some $i \neq$
$j$, the vertices $v_i$ and $v_j$ are in separate components of $H(i, j)$,
then we can pick the component containing $v_i$, exchange the
colors $c_i$ and $c_j$ in that component only. Now both $v_i$ and $v_j$
are colored with color $c_j$ so that $c_i$ can be used for $v$. 
We know that this has to happen for 2 of the colors since, if there is a path joining $v_1$ to $v_3$ in $H(1, 3)$ then there is no path joining $v_2$ to $v_4$ in $H(2, 4)$. □