Lesson 4 - Paths and Cycles

Walk, Trail, Path
length= number of edges
Connected graph, components
circuit=closed trail, cycle=closed path
Thm 3: u, v-walk implies u, v-path.
Thm 4: G is bipartite if and only if every cycle is even.
Eulerian Circuit, Eulerian graph, Eulerian trail, Semi-Eulerian graph
Konigsberg Bridges
Thm 5: Euler's Theorem
Cor 5.1: A connected graph G is semi-Eulerian iff there are exactly 2 vertices of odd degree.
Complexity of problem to decide if G is Eulerian.
Hamiltonian cycle, Hamiltonian graph
Hamiltonian path, Semi-Hamiltonian graph
Knight's Tour
Thm 6: Dirac's Theorem. Let G be a simple graph on n vertices with $n \geq 3$ vertices, $degv \geq n/2$, for all $v \in V(G)$ implies G is Hamiltonian.
Thm 7: Bondy and Chvatal's Theorem. Let G be a simple graph. If u, v are distinct nonadjacent vertices with $d(u) + d(v) \geq |V(G)|$, then G is Hamiltonian if and only if $G_u v$ is Hamiltonian.
Complexity of problem to decide if G is Hamiltonian.
Traveling Salesman Problem, Complexity.

Theorem 3. *Given a multi-graph G with distinct vertices u and v and a u, v-walk, W, there is a u, v-path using only edges within W.*

Proof: If the u, v-walk, W is not a path, then there is a vertex w repeated. We can remove the subsequence between the 2 occurrences of w and one copy of w, creating a u, v-walk that contains only edges from W. This process can be repeated until only a path remains. □

Proposition 1. *Given a multi-graph G and a closed walk, W in G, there is a cycle in G using only edges from W.*

Proof: In the closed walk W, we may consider there to be no particular starting vertex or ending vertex. If the closed walk, W is not a cycle, there is some vertex, w, that is repeated. We can remove the subsequence between the 2 occurrences of w and one copy of w, creating a closed walk containing only edges of W. This process can be repeated until only a cycle remains. □

Exercise 1. *Let G be a multi-graph. Every odd length closed walk contains an odd length cycle.*

Proof: Consider the process used in the proof of Proposition 1. We can insure that each time we remove a portion of the circuit, we are removing a cycle. That is, if w is repeated more than 2 times, just remove a sequence of the entire closed walk between two consecutive copies of w.
If no cycle that we remove is odd in length, then when the process ends, there is an odd number of edges left and there is a cycle left. Hence, there is an odd cycle left in the end.

Theorem 4. *G is bipartite if and only if every cycle is even.*

Proof: Suppose G is bipartite with partite sets A and B. If $C : v_1, v_2, \ldots, v_k$ is a cycle in G then the vertices must

alternate between sets A and B. Wlog, assume $v_1 \in A$. Then vertices with odd subscripts must be in A and vertices with even subscripts must be in B. But $v_k \in B$, so k is even.
Now suppose all cycles are even. Let $v \in V(G)$ be any vertex. Let A be the collection of vertices v and any vertex of even distance from v. Let B be the other vertices of G, that is the ones of odd distance from v.
We first show that there could be no edges in A. Suppose $x, y \in A$ and $xy \in E(G)$. (It is possible that one of these vertices could be v itself.) Then there is an even length v, x-path, call it P_1 and there is an even length y, v-path, call it P_2. The new walk $P : P_1, xy, P_2$ must have odd length. It is an exercise to show that every odd length closed walk contains an odd length cycle. (See Exercise 1.) But this is a contradiction. So no edge xy exists.
Now suppose $x, y \in B$ and $xy \in E(G)$. Then there is an odd length v, x-path, call it P_1 and there is an odd length y, v-path, call it P_2. The new walk $P : P_1, xy, P_2$ must have odd length. Again, by the exercise we see that this is a contradiction. So no edge xy exists. $\square$

Theorem 5. *Euler's Theorem. For a connected multi-graph G, G is Eulerian if and only if every vertex has even degree.*

Proof: If G is Eulerian then there is an Euler circuit, P, in G. Every time a vertex is listed, that accounts for two edges adjacent to that vertex, the one before it in the list and the one after it in the list. This circuit uses every edge exactly once. So every edge is accounted for and there are no repeats. Thus every degree must be even.
Suppose every degree is even. We will show that there is an Euler circuit by induction on the number of edges in the graph.

The base case is for a graph G with two vertices with two edges between them. This graph is obviously Eulerian.

Now suppose we have a graph G on $m > 2$ edges. We start at an arbitrary vertex v and follow edges, arbitrarily selecting one after another until we return to v. Call this trail W. We know that we will return to v eventually because every time we encounter a vertex other than v we are listing one edge adjacent to it. There are an even number of edges adjacent to every vertex, so there will always be a suitable unused edge to list next. So this process will always lead us back to v.

Let E be the edges of W. The graph $G - E$ has components $C_1, C_2, \ldots, C_k$. These each satisfy the induction hypothesis: connected, less than m edges, and every degree is even. We know that every degree is even in $G - E$, because when we removed W, we removed an even number of edges from those vertices listed in the circuit. By induction, each circuit has an Eulerian circuit, call them $E_1, E_2, \ldots, E_k$.

Since G is connected, there is a vertex a_i in each component C_i on both W and E_i. Without loss of generality, assume that as we follow W, the vertices $a_1, a_2, \ldots, a_k$ are encountered in that order.

We describe an Euler circuit in G by starting at v follow W until reaching a_1, follow the entire E_1 ending back at a_1, follow W until reaching a_2, follow the entire E_2, ending back at a_2 and so on. End by following W until reaching a_k, follow the entire E_k, ending back at a_k, then finish off W, ending at v. □

Corollary 5.1. *A connected multi-graph G is semi-Eulerian if and only if there are exactly 2 vertices of odd degree.*

Proof:

(⇒)

If G is semi-Eulerian then there is an open Euler trail, P, in G. Suppose the trail begins at u_1 and ends at u_n. Except for the first listing of u_1 and the last listing of u_n, every time a vertex is listed, that accounts for two edges adjacent to that vertex, the one before it in the list and the one after it in the list. This circuit uses every edge exactly once. So every edge is accounted for and there are no repeats. Thus every degree must be even, except for u_1 and u_n which must be odd.
($\Leftarrow$)
Suppose u and v are the vertices of odd degree. Consider $G+uv$. This graph has all even degrees. By Theorem 5, G has an Eulerian circuit. This circuit uses the edge uv. Thus we have an Euler path in G when we omit the edge uv.$\square$

Theorem 6. *Dirac's Theorem. Let G be a simple graph on n vertices with $n \geq 3$ vertices, $deg(v) \geq n/2$, for all $v \in V(G)$ implies G is Hamiltonian.*

Proof: Among all graphs on n vertices with $n \geq 3$ vertices and $deg(v) \geq n/2$ for all v, choose G to be maximal in the sense that if you add any edges to G, then it would be Hamiltonian. Now select 2 nonadjacent vertices u and v in G. We know that there is a Hamiltonian cycle if the edge uv were to be added to G, therefore, there is a Hamiltonian Path $P : u = u_1, u_2, \ldots, u_n = v$ in G from u to v.
Claim: There must be an i such that $2 \leq i \leq n-2$ with v adjacent to u_i and u adjacent to u_{i+1}.
If not, then for $2 \leq j \leq n-2$ there are at least $n/2-1$ vertices u_j that v is adjacent to where u is not adjacent to u_{j+1}. (The minus 1 comes from the fact that v is adjacent to u_{n-1}.) But that gives $n/2-1$ vertices that u is not adjacent to among $u_3, \ldots u_{n-1}$. That leaves at most $n-3-(n/2-1)+1 = n/2-1$

that u could be adjacent to. (The plus 1 comes from the fact that u is adjacent to u_2.) This contradicts that $deg(u) \geq n/2$. Now that we have edges vu_i and uu_{i+1}, we see that there is a Hamiltonian circuit,

$$C : u = u_1, \ldots, u_i, u_n = v, u_{n-1}, \ldots, u_{j+1}, u_1 = u \square$$

Theorem 7. *Bondy and Chvatal's Theorem. Let G be a simple graph. If u, v are distinct nonadjacent vertices with $d(u) + d(v) \geq |V(G)|$, then G is Hamiltonian if and only if $G + uv$ is Hamiltonian.*

Proof: Clearly, if G is Hamiltonian then $G+uv$ is Hamiltonian. Now suppose $G + uv$ is Hamiltonian. We follow the method in the previous proof. We know that there is a Hamiltonian cycle if the edge uv were to be added to G, therefore, there is a Hamiltonian Path $P : u = u_1, u_2, \ldots, u_n = v$ in G from u to v.
There must be an i such that $1 < i < n - 1$ with v adjacent to u_i and u adjacent to u_{i+1}. If not, then for $2 \leq j \leq n - 2$ there are at least $deg(v) - 1$ vertices u_j that v is adjacent to where u is not adjacent to u_{j+1}. (The minus 1 comes from the fact that v is adjacent to u_{n-1}.) But that gives $deg(v) - 1$ vertices that u is not adjacent to among $u_3, \ldots u_{n-1}$. That leaves $n - 3 - (deg(v) - 1) = n - deg(v) - 2$ among those. Counting u_2, we have that $deg(u) \leq n - deg(v) - 1$. This contradicts that $deg(u) + deg(v) \geq n$.
Now that we have edges vu_i and uu_{i+1}, we see that there is a Hamiltonian circuit,

$$C : u = u_1, \ldots, u_i, u_n = v, u_{n-1}, \ldots, u_{j+1}, u_1 = u \square$$