ERT on Maximum degree $D$

Theorem 9 states that if $G$ is not complete and not an odd cycle then $G$ is $\Delta(G)$ choosable. This follows as well from the 1979 paper of Erdös, Rubin, and Taylor.

**Defn 13** Let $D$ be the function on the vertices of $G$, $D(v) = \deg(v)$. We say $G$ is $D$-choosable if for every list assignment, $L$, such that for all $v$, $|L(v)| = D(v)$, $G$ is $L$-choosable.

**Defn 14** A $\theta$ graph consists of two distinct vertices $i$ and $j$ together with three paths which are vertex disjoint except that each path has $i$ at one end and $j$ at the other end.

Examples: $\theta_{1,2,2}, \theta_{1,2,3}, \theta_{1,3,3}, \theta_{2,2,2}, \theta_{2,2,4}, \theta_{2,2,6}$.

**Maximum degree $D$**

**Defn 15** Suppose $G$ and $H$ are two graphs, take any nodes $i$ of $G$ and $j$ of $H$, merge them into a single node $\widehat{ij}$ to produce a new graph $G\widehat{ij}H$. Then let $\text{NON-D}$ be the set containing all complete graphs, all odd cycles, and whenever $G, H \in \text{NON-D}$, put $G\widehat{ij}H$ in $\text{NON-D}$.

A Typical member of $\text{NON-D}$:

**Theorem 10** (ERT) If $G$ is connected then $G$ is not $D$-choosable $\iff G \in \text{NON-D}$. Equivalently, $G$ is $D$-choosable $\iff G$ contains an induced even cycle or an induced theta-graph.
Corollary 10.1  If $G$ is connected, not $K_n$ and not an odd cycle, then $\text{ch}(G) \leq \Delta(G)$.

Proof of Corollary 10.1: Suppose $G$ is not $\Delta(G)$ regular. Since $G$ is connected, we can pick a vertex of degree $b < \Delta(G)$ to be the root vertex of a spanning tree of $G$. Order the vertices from left to right so that for each vertex other than the root, its parent in the spanning tree is to its right. Apply the greedy algorithm starting with the left most vertex. Thus, if $G$ is not $\Delta(G)$-regular, then $\text{ch}(G) \leq \Delta(G)$.

Now suppose there is a $G$ such that $\text{ch}(G) \geq \Delta(G) + 1$. Then $G$ is not $D$-choosable. By Theorem 10 $G$ is in $\text{NON-D}$. But the only regular graphs in $\text{NON-D}$ are the odd cycles and complete graphs. \(\bigcirc\)

The List Coloring Conjecture

Defn 16  The line graph of a graph $G$, denoted $L(G)$, is the graph whose vertices are the edges of $G$. Two vertices in $L(G)$ form an edge if and only if their corresponding edges in $G$ share an endpoint.

Notation: $\chi(G)$, $\chi^e(G)$, $\text{ch}(G) \equiv \chi_e(G)$, $\chi^e_e(G)$.

In 1985, Vizing, Albertson, Collins, Tucker, Gupta, Bollobás and Harris, conjectured that for all $G$, the choice number of $L(G)$ is equal to the chromatic number of $L(G)$.

Conj 1  For all $G$, $\chi(L(G)) = \chi^e_e(L(G))$ or equivalently, $\chi^e(G) = \chi^e_e(G)$.

In 1993, Galvin proved the conjecture for bipartite graphs. This fact was then used to solve a well known conjecture of Dinitz (1979). He used algebraic techniques of Alon & Tarsi.

In 1996, Tomaž Slivnik published A short proof of Galvin’s theorem on the list-chromatic index of a bipartite multigraph.

DINITZ’s Conjecture of 1979

Theorem 11  Given an $n \times n$ array, and for each $i, j$, a set, $A_{i,j}$ of $n$ numbers, there exists an assignment of a number to each position where for position $i, j$ the number comes from set $A_{i,j}$ and there are no repeats in any row or in any column.
The choice number of the line graph of $K_{n,n}$ is at most $n$.

We have a theorem that says if the $G$ is bipartite then $\chi^e(G) = \Delta(G)$.
Notice $\Delta(K_{n,n}) = n$, which implies that $\chi^e(K_{n,n}) = n$.
Thus it would suffice to prove that if $G$ is bipartite then $\chi^e(G) = \Delta(G)$.

Colorings and Orientations of Graphs
by N. Alon and M. Tarsi (1992)

The abstract of their paper reads: Bounds for the chromatic number and for some related parameters of a graph are obtained by applying algebraic techniques. In particular, the following result is proved: If $G$ is a directed graph with maximum outdegree $d$, and if the number of Eulerian subgraphs of $G$ with an even number of edges differs from the number of Eulerian subgraphs with an odd number of edges then for any assignment of a set $S(v)$ of $d+1$ colors for each vertex $v$ of $G$ there is a legal vertex-coloring of $G$ assigning to each vertex $v$ a color from $S(v)$.

They use:

**Theorem 12** Let $P = P(x_1, x_2, \ldots, x_n)$ be a polynomial in $n$ variables over the ring of integers $\mathbb{Z}$. Suppose that for $1 \leq i \leq n$ the degree of $P$ as a polynomial in $x_i$ is at most $d_i$ and let $S_i \subseteq \mathbb{Z}$ be a set of $d_i+1$ distinct integers. If $P(x_1, x_2, \ldots, x_n) = 0$ for all $n$-tuples $(x_1, \ldots, x_n) \in S_1 \times S_2 \times \cdots \times S_n$ then $P \equiv 0$.

Galvin’s Proof of Dinitz’s Conj. 1993
Some background about digraphs

**Defn 17** For this talk, a digraph is an orientation of a simple graph. A kernel $K$ of a digraph $D$ is a subset of the vertices so that if $u, v \in K$ then neither $(u, v)$ nor $(v, u)$ is an edge in $D$ and if $v \notin K$, then $\exists w \in K$ such that $(v, w) \in E(D)$. A $u, v$-path in a digraph is a sequence of vertices that forms a path in the underlying graph such that the edges correspond to arcs with the head immediately following the tail in the path. Similarly, consider a cycle to be oriented. A digraph is weakly connected if its underlying graph is connected. A digraph is strongly connected if for each ordered pair $u, v$ of vertices, there is path from $u$ to $v$. The strong components are its maximal strong subgraphs.
Example: $D = (V, E)$, $V = \{a, b, c, d, e, f, g\}$,

$E = \{(a, b), (a, g), (b, c), (c, g), (d, c), (d, b), (d, g), (e, d), (e, f),
(e, g), (f, a), (f, g)\}$.

$K_1 = \{a, c, e\}$ is not a kernel. $K_2 = \{b, g\}$ is a kernel.

Galvin’s Proof of Dinitz’s Conj. 1993
Some background about digraphs

**Theorem 13** (Richardson, 1953) Every digraph $D$ having no odd cycle has a kernel.

**Proof:**

**Case 1:** $D$ is strongly connected.

Given an arbitrary vertex $y \in V(D)$, let $K$ be the set of vertices with even distance to $y$.

First suppose $K$ is not an independent set. Suppose $(u, v)$ is an edge in $< K >$. There is a $u, y$-path $P$ of even length and a $v, y$-path $P'$ of even length. Adding $uv$ to the start of $P'$ yields a $u, y$-walk $W$ of odd length. Because $D$ is strong, $D$ has a $y, u$-path $Q$. Thus, either $Q$ followed by $P$ or $Q$ followed by $W$ yields an odd closed walk in $D$ which would contain an odd closed cycle. This is a contradiction.

Next suppose that $v \notin K$, then, since the distance from $v$ to $y$ must be odd, the next vertex on the shortest $v, y$-path must be in $K$.

Richardson, 1953

**Case 2:** $D$ is not strongly connected.

We prove by induction on $|V(D)|$. If $|V(D)| = 1$, the kernel is the whole graph. Let $|V(D)| > 1$ and suppose the theorem is true for all digraphs on fewer vertices. Let $D'$ be a strong component such that there is no edge from a vertex of $D'$ to a vertex not in $D'$.

(It is an exercise to show that in every digraph, some strong component has no entering edges and some strong component has no exiting edges.)

We know from Case 1 that $D'$ has a kernel. Let $K'$ be a kernel of $D'$.

Let $D''$ be the sub-digraph obtained from $D$ by deleting $D'$ and all the predecessors of $K'$. (A vertex $v \notin K'$ is a predecessor of $K'$ if there exists a vertex $w \in K'$ such that $(v, w)$ is an arc of $D$.)
By the induction hypothesis, $D''$ has a kernel. Let $K''$ be a kernel of $D''$. We claim that $K = K' \cup K''$ is a kernel of $D$. There can be no edges in $K$ by the choice of $D'$ and the fact that we removed all predecessors of $K'$. Every vertex in $V(D) \setminus K$ is either in $D'$ or a predecessor of $K'$ or in $D''$ and so has an arc to a vertex of $K$.

Galvin’s Proof of Dinitz’s Conj. 1993

Some background about digraphs

The following Theorem was proved by Bondy, Boppana and Siegal.

**Theorem 14** (BBS) Suppose every induced sub-digraph of a digraph $D$ has a kernel. Then $D$ is $f$-choosable, where for each $v$, $f(v) = 1 + \deg^+(v)$.

**Proof:** Suppose every induced sub-digraph of $D$ has a kernel. We will use induction on $|V(D)|$. The theorem is clearly true if $|V(D)| = 1$. Suppose $|V(D)| > 1$ and the theorem is true for smaller such digraphs. Let $L$ be a list assignment of $D$ with $|L(v)| \geq f(v)$, for all $v$.

We arbitrarily choose a color from some list, $x \in L(v)$, for some $v \in V(D)$. Let $D_1$ be the sub-digraph induced on $X = \{v \in V(D) : x \in L(v)\}$. Let $K$ be a kernel of $D_1$.

Define $D_2 = D \setminus K$. If $v \in V(D_2)$, define $L_2(v) = L(v) - x$. The list assignment $L_2$ has the property that for all $v \in D_2$, $|L_2(v)| \geq \deg^+_{D_2} + 1$ so by induction $D_2$ has an $L_2$ list-coloring. We can color the vertices in $K$ by the color $x$. \( \diamond \)

Galvin’s Proof of Dinitz’s Conj. 1993

**Defn 18** A source is a vertex $v$ such that $\deg^-(v) = 0$. A sink is a vertex $v$ such that $\deg^+(v) = 0$. A tournament is an orientation of $K_n$. A tournament is transitive if the vertices can be labelled $v_1, v_2, \ldots, v_n$ so that for all $i < j$, $(v_i, v_j)$ is an arc. An orientation $D$ of a graph $G$ is called normal if every clique in $G$ is a transitive tournament in $D$.

The following theorem was proved by Maffray (1992).

**Theorem 15** Every normal orientation of the line graph $L$ of a bipartite graph $G$ has a kernel.

**Proof:** We use induction on $m = |E(G)|$.

Let $(A, B)$ be the bipartition of $G$. Let $D$ be a normal orientation of $L$. For a vertex $x$ of $G$, let $T_x$ be the tournament on the clique in $D$ corresponding to the edges incident to $x$ in $G$.  

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Case 1 There exists \( a \in A, b \in B \), such that \( e_1 = (a, b) \) is a sink in \( T_a \), and the source \( e_2 \) of the tournament \( T_b \) on the b-clique does not equal \( e_1 \).

Consider \( G' = G - e_2 \) and \( D' \) the restriction of \( D \) to \( L(G') \). \( D' \) is an orientation of a bipartite graph and so by induction, \( D' \) has a kernel \( K \). We claim that \( K \) is a kernel of \( D \). We need only show that there is some \( e_3 \) in \( K \) such that \( (e_2, e_3) \) is an arc in \( D \). We know that \( (e_2, e_1) \) in \( D \) so if \( e_1 \in K \), we are fine.

If \( e_1 \notin K \) we choose \( e_3 \in K \) such that \( (e_1, e_3) \) is an arc in \( D \). We know that \( e_3 \notin T_a \) since \( e_1 \) is the sink in \( T_1 \). So, \( e_3 \in T_b \) and since \( e_2 \) is the source in \( T_b \), we are done.

Case 2 For all \( a \in A \) and \( b \in B \), if \( (a, b) \) is a sink in \( T_a \) then \( (a, b) \) is also the source in \( T_b \).

In this case, let \( K = \{ e_a : a \in A, e_a \text{ is a sink in } T_a \} \). Suppose \( e_a \) and \( e_{a'} \) are two vertices in \( K \). If \( e_a \) and \( e_{a'} \) are incident to a vertex \( b \) in \( B \) then they are both in \( T_b \). But then due to the case we are in, they would both have to be the source in \( T_b \). There is only one source. Thus, there is no edge between them. Now let \( e \in V(D) \setminus K \). Then \( e = (a, b) \) or \( e = (b, a) \) for some \( a \in A, b \in B \). So \( e \) is in \( T_a, \) whose sink is in \( K \). Galvin’s Proof of Dinitz’s Conj. 1993

Theorem 16 If \( G \) is bipartite then \( \chi_L^e(G) = \Delta(G) \).

Proof: Let \( (A, B) \) be the bipartition of \( G \). Let \( c \) be a proper vertex coloring of \( L = L(G) \) with \( \{1, 2, \ldots, \chi(L) = \Delta(G)\} \).

Orient \( L \) to form \( D \) as follows: If \( \{e_1, e_2\} \in E(L) \) with \( c(e_1) < c(e_2) \) then we orient the edge as \( (e_1, e_2) \).

We see that \( D \) is a normal orientation of \( L \) as follows. Every clique is contained in \( T_a \) for some \( a \in A \) or \( T_b \) for some \( b \in B \). For a given \( a \in A \), the coloring of the edges in \( T_a \) has some linear order.

By Maffray’s Theorem, \( D \) has a kernel. We can assume as well that every sub-digraph of \( D \) has a kernel. Consider a vertex \( e \in V(D) \) with its out neighbors, they all have different colors in our coloring \( c \) of \( L \). Thus \( \deg^+(e) + 1 \leq \chi(L) \). Now by BBS, \( D \) and hence \( L \) is \( \chi(G) \) choosable.

Note: We never used Alon and Tarsi.