List Coloring Graphs

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CHROMATIC NUMBER

Defn 1  A $k$-coloring of a graph $G$ is a function $c : V(G) \rightarrow \{1, 2, \ldots k\}$. A proper $k$-coloring of a graph $G$ is a coloring of $G$ with $k$ colors so that no 2 distinct adjacent vertices are the same color.

The chromatic number of $G$, $\chi(G)$, is the smallest $k$ such that a proper $k$-coloring of $G$ exists.

LIST COLORINGS AND CHOICE NUMBER

Defn 2  A $k$-list assignment, $L$, is an assignment of sets (called lists) to the vertices

$$v \mapsto L(v)$$

such that

$$|L(v)| \geq k,$$

for all vertices $v$.

An $L$-list coloring is a coloring such that the color assigned to $v$ is in $L(v)$ for all vertices $v$.

A proper $L$-list coloring is an $L$-list coloring which is proper.

If $G$ is such that a proper $L$-list coloring exists for all possible $k$-list assignments $L$, we say that $G$ is $k$ choosable. The smallest $k$ for which $G$ is $k$ choosable is the choice number of $G$, denoted $ch(G)$. 
EXAMPLES OF LIST COLORING

FACTS ABOUT LIST COLORING

**Theorem 1** For all graphs on $n$ vertices,

$$\chi(G) \leq \text{ch}(G) \leq \chi(G) \ln(n)$$

First, we will show $\text{ch}(G) > \chi(G) - 1$. This gives the lower bound.
Set $k = \chi(G)$ and assign the list $\{1, 2, \ldots, k-1\}$ to all vertices.
This is an example of a $k-1$-list assignment $L$ which cannot be properly colorable.
Thus $\text{ch}(G) > \chi(G) - 1$.

FACTS ABOUT LIST COLORING

Next, we show $\text{ch}(G) \leq \chi(G) \ln |V(G)|$.

A probabilistic argument.

Color $G$ with $s = \chi(G)$ colors,

Color classes: $C_1, C_2, \ldots, C_s$

Suppose $k = \chi(G) \ln n$, where $n = |V(G)|$.

Assume $G$ has a $k$-list assignment $L$.

The sample space is the set of all partitions of $\bigcup_{u \in V(G)} L(v)$ into at most $s$ parts. A typical partition is $P: P_1, P_2, \ldots, P_s$ and

For $c \in \bigcup_{u \in V(G)} L(v)$, Prob $(c \in P_i) = \frac{1}{s}$

Want $\forall i \in [s], \forall v \in C_i, L(v) \cap P_i \neq \emptyset$.

Prob $(\exists i \in [s], \exists v \in C_i, L(v) \cap P_i = \emptyset)$

$\leq \sum_{v \in V(G)} \left(\frac{1}{s}\right)^k = n((1 - \frac{1}{s})^s)^\frac{k}{s} < ne^{-\frac{k}{s}}.$
We see that \( s \ln n = k \)

\[
\ln n = \frac{k}{s}
\]

\[
n = e^{\frac{k}{s}}
\]

\[
ne^{-\frac{k}{s}} = 1
\]

And so \( \text{Prob} (\exists i \in [s], \exists v \in C_i, L(v) \cap P_i = \emptyset) < ne^{-\frac{k}{s}} = 1 \)

Therefore, the probability

\[
\forall i \in [s], \forall v \in C_i, L(v) \cap P_i \neq \emptyset > 0.
\]

So there exists such a partition.

**Planar Graphs**

**Defn 3** *A graph is planar if it can be drawn in the plane with no edge crossings.*

**Defn 4** *A graph is bipartite if and only if its chromatic number is 2.*

Remember theorem: If \( G \) is planar and bipartite then \( m \leq 2n - 4 \) where \( m \) is the number of edges and \( n \) is the number of vertices.

**Planar Graphs**

Let go over some terms:
Defn 5 Face or region. Interior face. Exterior or unbounded face.

Defn 6 The length of a face. Total length of closed walk(s) in \( G \) bounding the face.

EXAMPLES

Defn 7 A connected plane embedding of a graph has a unique closed walk bounding the exterior region. We call that walk the outercircuit. (Note that for a closed walk, trail, or path, the starting point is irrelevant, so technically we could say a closed walk is an equivalence class of all closed walks following the same order of vertices and edges. Also, even though a circuit is a closed trail and we are talking about a closed walk here, we still call it the outercircuit.)

Defn 8 Outerplanar graph. All vertices are on the boundary of the exterior region. Note that the boundary is an outer circuit if the graph is connected.

Defn 9 Given a cycle \( C \) in a plane embedding of \( G \), \( C \) divides the plane into 2 regions, one containing the unbounded face. The exterior of \( C \), \( \text{ext}(C) \), is the maximum subgraph of \( G \) embedded in region containing the unbounded face. The interior of \( C \), \( \text{int}(C) \), is the maximum subgraph of \( G \) embedded in the other region. Neither ones contain any vertices of \( C \) itself.

Planar Graphs

If \( A \subset V(G) \), we use the notation \(< A >\) for the induced graph on the vertices in \( A \). If we want to indicate the induced graph on the vertices in the exterior of \( C \) together with the vertices of \( C \), we could write: \(< V(\text{ext}(C)) \cup V(C) >\).

Defn 10 A chord of a cycle is an edge whose endpoints are both in the cycle, but is not itself a cycle edge. (This can be applied to any cycle - planar or not.)

Chronology:
1. Chromatic Number

   (a) Heawood 1890, 5-color theorem
       For planar graphs
   (b) Grötzsch 1959, 3-color theorem
       For planar graphs of girth 4
   (c) Grünbaum 1962, 3-color theorem
       For planar graphs with at most 3 3-cycles
   (d) Appel, & Haken 1976, 4-color theorem
       For planar graphs
   (e) N. Robertson, D. P. Sanders, P. D. Seymour and R. Thomas, 1996,
       4-color theorem
       For planar graphs

2. List Coloring and Choice Number

   (a) Alon & Tarsi 1992, 3-choosable
       For planar and bipartite
   (b) Thomassen 1994, 5-choosable
       For planar graphs
       Implies (1a)
   (c) Thomassen 1995, 3-choosable,
       For planar graphs of girth 5
       Implies (1b)
   (d) Thomassen 2003, 3-choosable,
       For planar graphs of girth 5
       A short proof

Thomassen 5-color
1994 Thomassen

**Theorem 2** If \( G \) is planar then the choice number is at most 5.

He proved something stronger:

**Theorem 3** Let \( G \) be a connected planar with outercircuit \( C = (v_1, v_2, \ldots, v_k) \), and let \( L \) be a list assignment such that for \( v \in V(C) \), \( |L(v)| \geq 3 \) otherwise \( |L(v)| \geq 5 \). For any precoloring, \( c \), of the vertices \( v_1 \) and \( v_2 \), \( c \) can be extended to an \( L \)-coloring of \( G \).

**Corollary 3.1** If \( G \) is an outerplanar graph then \( ch(G) \leq 3 \).

Notice that Theorem 3 implies Theorem 2.

**Proof**

We can assume \( G \) is inner triangulated.

Suppose \( G \) has a cut vertex....

Now we can assume the outercircuit \( C \) is a cycle.

If \( C \) has a chord......

If \( C \) does not have a chord.....

Thomassen’s implies Grotzsch

(A) Grotzsch: Every planar graph \( G \) of girth at least 4 is 3-colorable. Moreover, if \( G \) has an outer cycle of length 4 or 5 then any 3-coloring of the outer cycle can be extended to a 3-coloring of \( G \).

(B) Grotzsch’s girth 5 version: Every planar graph \( G \) of girth at least 5 is 3-colorable. Moreover, if \( G \) has an outer cycle of length 5 then any 3-coloring of the outer cycle can be extended to a 3-coloring of \( G \).
We will show (B) $\iff$ (A).

**Thomassen’s Long proof**

**Theorem 4** Let $G$ be a planar graph of girth at least 5. Let $A$ be a set of vertices in $G$ such that each vertex of $A$ is on the outer cycle. Assume that either

(i) $G(A)$ has no edge or

(ii) $G(A)$ has precisely one edge $xy$ and $G$ has no 2-path from $x$ to a vertex of $A$.

Assume that $L$ is a color assignment such that $|L(v)| \geq 2$ for each vertex in $G$ and $|L(v)| \geq 3$ for each vertex in $V(G) \setminus A$. Let $u, w$ be any adjacent vertices in $G$ both on the outer face boundary and let $c(u), c(w)$ be distinct colors in $L(u)$ and $L(w)$ respectively. Then $c$ can be extended to a list coloring of $G$.

**Thomassen’s Short proof**

**Theorem 5** Let $G$ be a plane graph of girth at least 5. Let $c$ be a 3-coloring of a path or cycle $P : v_1, v_2, \ldots, v_q$, $1 \leq q \leq 6$ such that all vertices of $P$ are on the outer face boundary.

For all $v \in V(G)$, let $L(v)$ be its list of colors. If $v \in P$ then $L(v) = \{c(v)\}$. Otherwise $|L(v)| \geq 2$. If $v$ is not on the outer face boundary then $|L(v)| = 3$.

There are no edges joining vertices whose lists have at most 2 colors, except the edges of $P$.

Then $c$ can be extended to an $L$-coloring of $G$.

**Grotzsch’s girth 5 version**

Notice that both of these imply Grotzsch’s girth 5 version.
We will show this later.

Grötzsch (B) ⇒ (A)

(A) Grotzsch: Every planar graph $G$ of girth at least 4 is 3-colorable. Moreover, if $G$ has an outer cycle of length 4 or 5 then any 3-coloring of the outer cycle can be extended to a 3-coloring of $G$.

(B) Grotzsch’s girth 5 version: Every planar graph $G$ of girth at least 5 is 3-colorable. Moreover, if $G$ has an outer cycle of length 5 then any 3-coloring of the outer cycle can be extended to a 3-coloring of $G$.

Use (B) to prove (A).

Grötzsch (B) ⇒ (A)

Proof by induction on $|V(G)|$.
If $G$ has no 4 cycles then by (B) we are done.
Assume $G$ has 4 cycles.
If $G$ has a vertex $v$ of degree at most 2, color $G - v$ by induction, and color $v$ afterward with a color not used on either of its neighbors.
So assume all degrees are at least 3.
If $G$ is disconnected, use induction on each component.
If $G$ has a cut vertex $v$, such that $G - v$ has components, $C_1, C_2, \ldots, C_k$, color each of $C_i + v$ using induction then permute the colors if necessary so that $v$ is colored the same in each.

Grötzsch (B) ⇒ (A)

Now we assume that $G$ is 2-connected. So that every facial walk is a cycle.
Suppose the length of $C$ is greater than 5. Then it is not precolored. We
find a facial cycle of length at most 5 and make that the outer cycle and precolor it.

We know there is a cycle of length 4 but need a facial cycle of length 4 or 5.

We know such a facial cycle exists by Euler’s formula and the fact that we know for all $v$, the degree of $v$ is at least 3, as shown on next page

Grötzsch (B) ⇒ (A)

Let $f$ be the number of faces, $n$ the number of vertices, and $e$ the number of edges.

$$\sum_{i=1}^{n} \deg(v_i) = 2e, \ 3n \leq 2e$$

$$n - e + f = 2, \ n = 2 - f + e$$

$$3(2 - f + e) \leq 2e, \ e \leq 3f - 6$$

But if $\forall i, \deg(f_i) \geq 6$ then

$$2e = \sum_{i=1}^{f} \deg(f_i) \geq 6f$$

and $e \geq 3f$. This is a contradiction. Grötzsch (B) ⇒ (A)

So we assume we have outer cycle $C$ of length 4 or 5 and it is precolored.

**Defn 11** A separating cycle is a cycle $C$ in a plane embedding of a graph where both $ext(C)$ and $int(C)$ are non-empty.

If $G$ has a separating 4 or 5 cycle $C'$. We color $ext(C') \cup C'$ first using induction then color $int(C') \cup C'$ by induction.

If $G$ has a vertex joined to two vertices of $C$ we precolor that vertex in $int(C)$ and use induction on the 2 parts.
Grötzsch (B) ⇒ (A)

If \( G \) has a facial 4-cycle distinct from \( C \), identify 2 opposite vertices and use induction to color the resulting graph, then reverse the process and the same color is valid for both of the original vertices.

**Claim 1** If identifying the 2 opposite vertices causes a 3-cycle, then identifying the other 2 opposite vertices of the 4-cycle will not cause a 3-cycle.

**Proof of Claim:** Suppose the 4-cycle is \( u_1, u_2, u_3, u_4, u_1 \). If identifying \( u_1 \) and \( u_3 \) gives a 3-cycle it must involve the new vertex \( U \). Suppose then it is \( U, x_1, x_2, U \). It is clear that \( x_1, x_2, u_2, u_4 \) are 4 distinct vertices and both \( x_1 \) and \( x_2 \) are not adjacent to either of \( u_1 \) or \( u_3 \) since there are no 3-cycles in \( G \).

So without loss of generality, we must have in the original graph, the path: \( u_1, x_1, x_2, u_3 \). If we had instead joined together \( u_2 \) and \( u_4 \), we would be able to conclude that there is a path \( u_2, y_1, y_2, u_4 \). The vertices \( x_1, x_2, y_1 \) and \( y_2 \) are 4 distinct vertices. If any two were the same, there would be a triangle.

But now notice that we have a path from \( u_1 \) to \( u_3 \) and a path from \( u_2 \) to \( u_4 \) which are disjoint. Thus one must go through the interior of the cycle: \( u_1, u_2, u_3, u_4, u_1 \). But this cycle was chosen as a "facial" cycle, meaning its interior is empty. ♦

Notice that we cannot simply add 5th vertices to all the 4 cycles of \( G \) and apply (B).

Otherwise, we have no separating 4 cycle and no facial 4 cycle so, we can assume that \( C \) is the only 4-cycle. We CAN insert a new vertex on \( C \) and precolor it. Now we apply (B)