Mth 307 Mathematical Rigor

Exam 3 - Fall 2005

Answer any 6 of the following 7 questions. You may earn extra credit by answering them all.

(1) Let $g : \mathbb{R} \times \mathbb{R} \to \{x \in \mathbb{R} : x \geq 0\}$ be the function $g(a, b) = |a \cdot b|$.
  (a) Prove or disprove the statement: g is injective.
  Sol: g is not injective.
  Let $(A_1, b_1) = (1, 0)$ and $(a_2, b_2) = (0, 1)$.
  We see that $g(a_1, b_1) = g(1, 0) = |1 \cdot 0| = 0$ and $g(a_2, b_2) = g(0, 1) = |0 \cdot 1| = 0$, but $(a_1, b_1) \neq (a_2, b_2)$.
  (b) Prove or disprove the statement: g is surjective.
  Sol: g is surjective.
  Let y be an element of the codain, $\{x \in \mathbb{R} : x \geq 0\}$. The $(1, y)$ is an element of the domain, $\mathbb{R} \times \mathbb{R}$ and $g(1, y) = |1 \cdot y| = |y| = y$, since $y \geq 0$.

(2) Complete all steps necessary to prove that the odd natural numbers are countable.
Sol: Let $\mathcal{O}$ be the odd natural numbers. $\mathcal{O}$ is countable if $\mathcal{O}$ is finite or countably infinite. We will show that $\mathcal{O}$ is countably infinite by describing a bijection from $\mathbb{N}$ to $\mathcal{O}$.

Let $f : \mathbb{N} \to \mathcal{O}$ be defined by $f(n) = 2n - 1$.

Let $n_1, n_2 \in \mathbb{N}$ and assume $f(n_1) = f(n_2)$. Then $2n_1 - 1 = 2n_2 - 1$, so $n_1 = n_2$. Thus f is one-to-one.

Let $x \in \mathcal{O}$. Then $x + 1$ is even, so $(x + 1)/2 \in \mathbb{Z}$ and $x \geq 1$ implies that $x + 1 \geq 2$ so $(x + 1)/2 \geq 1$. We have that $(x + 1)/2 \in \mathbb{N}$ which is the domain. We have

$$f(\frac{x+1}{2}) = 2(\frac{x+1}{2}) - 1 = x + 1 - 1 = x.$$

Thus, f is onto.

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} |x - 2|, & x \geq 2 \\ |x| - 2, & x \leq 7 \end{cases}$$

Prove or disprove the statement: f is well-defined.
Sol: It is clear that each element of the domain maps to exactly one element of the range except possibly when $2 \leq x \leq 7$.

If $2 \leq x \leq 7$ then $x \geq 0$ and $x - 2 \geq 0$, so, $|x - 2| = x - 2$ and $|x| - 2 = x - 2$. So we see that even if x is between 2 and 7, there is only one function value for x. Thus f is well-defined.

(4) Let $n, k \in \mathbb{N}$ with $k \leq n$. Prove that

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.$$

Proof. Let S be the set of k-element subsets of the set $[n+1] = \{1, 2, \ldots, n+1\}$. We compute $|S|$ in two ways.

The first way is that by that by the definition of collections, $C(n+1,k)$, $\binom{n+1}{k}$ is the number of k-element subsets of an $n+1$-element set.

On the other hand, we partition S into two blocks. Block 1, B_1 will contain the subsets that have the element $n+1$ in them and block 2, B_2 will be the subsets that do not have the element $n+1$ in them. Then $|B_1|$ is $\binom{n}{k-1}$ since we have to choose $k-1$ other elements from the n-element set $\{1,2,\ldots,n\}$ and $|B_2|$ is $\binom{n}{k}$ since we have to choose k other elements from the n-element set $\{1,2,\ldots,n\}$.

Hence, $|S| = |B_1| + |B_2| = \binom{n}{k-1} + \binom{n}{k}$.

So $\binom{n+1}{k}$ must be equal to $\binom{n}{k-1} + \binom{n}{k}$. □

(5) Suppose we have a special alphabet of 20 letters. We will call a special word any sequence of 5 letters with at most 2 repeated. How many special words are there using this special alphabet.

Sol:

$$_{20}P_5 +_{20} C_4 \cdot 4 \cdot \frac{5!}{2!} = 1,860,480 + 1,162,800 = 3,023,280$$

Since there are $_{20}P_5$ words with no repeated letters.

There are $_{20}C_4$ ways to choose 4 letters and 4 ways to select one of those to be repeated and $\frac{5!}{2!}$ ways to arrange the 5 letters once they are chosen, like in "greet".

(6) For each of the following graphs, use a theorem to determine if it is Eulerian or not. Explain. If it is Eulerian, show an Euler Circuit in the graph.

`Just ask me about this one if you don't understand it.`

(7) Let $n \in \mathbb{N}$, $\mathcal{A}$ be the odd sized subsets of $[n]$ and let $\mathcal{B}$ be the even sized subsets of $[n]$. Use the Binomial Theorem to show that $|\mathcal{A}| = |\mathcal{B}|$.

Hint, in the case when $n = 6$, this is the same thing as showing

$$\binom{6}{0} + \binom{6}{2} + \binom{6}{4} + \binom{6}{6} = \binom{6}{1} + \binom{6}{3} + \binom{6}{5}$$

which is true if and only if

$$\binom{6}{0} - \binom{6}{1} + \binom{6}{2} - \binom{6}{3} + \binom{6}{4} - \binom{6}{5} + \binom{6}{6} = 0.$$

Sol: By the Binomial Theorem:

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n$$

So we have

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} = (-1+1)^n = 0^n = 0$$

Thus,

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots \pm \binom{n}{n} = 0.$$

So, when n is even,

$$\binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{n} = \binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{n-1}.$$

When n is odd,

$$\binom{n}{0}+\binom{n}{2}+\cdots+\binom{n}{n-1}=\binom{n}{1}+\binom{n}{3}+\cdots+\binom{n}{n}.$$

In both equations the sum on the left represents $|\mathcal{B}|$ and the sum on the right represents $|\mathcal{A}|$. Thus $|\mathcal{A}| = |\mathcal{B}|$.