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MTH 307 2.1.2

A point \((m, n)\) in the plane \(\mathbb{R}^2\) is a **lattice point** if both coordinates \(m\) and \(n\) are integers. Prove that the number of lattice points inside any circle centered at the origin is a number of form \(4k + 1\) for some integer \(k\).

Let \(L_i\) be the set of all lattice points in the plane \(\mathbb{R}^2\), within a single quadrant,

\[
L_i = \{(m, n) \mid m^2 + n^2 \leq r^2, (m, n) \in Q_i, m \neq 0, n \neq 0, m, n, r \in \mathbb{Z}\}
\]

Since \(L_i\) describes only the lattice points within the circle in the \(i\)th quadrant, and not the lattice points where \(m\) or \(n\) are equal to zero, we need another set describing the lattice points where \(m\) or \(n\) are equal to zero:

\[
K = \{(m, n) \mid m^2 + n^2 \leq r^2, (m = 0) \vee (n = 0), m, n, r \in \mathbb{Z}\}.
\]

\(K\) describes the set of all lattice points within the circle where \(m\) or \(n\) are equal to zero. The sets \(L_i\) and \(K\) are mutually exclusive sets, so the union of \(L_i\) and \(K\) for all \(i\) describe the set of all lattice points in the plane \(\mathbb{R}^2\) within the circle.

**lattice points** = \(K \cup_i L_i\)

Thus, the number of lattice points in the circle is equal to:

\[
\# \text{ of lattice points} = |K \cup_i L_i|
\]

\[
= |K| + |\cup_i L_i|
\]

By the definition of \(K\), we know that the origin is included in \(K\). We will denote the origin by \(O\). The set \(K \setminus O\) is the set of all lattice points in the circle such that \(m\) or \(n\) is equal to 0, but not both.
\[ K \setminus O = \{(m, n) | m^2 + n^2 \leq r^2, (m = 0) \text{XOR}(n = 0), m, n, r \in \mathbb{Z}\}. \]

The set \( K \setminus O \) maintains symmetry, as it extends along both axes in the Cartesian plane of \( \mathbb{R}^2 \). The number of lattice points in the circle about the origin is now:

\[ = |K \setminus O| + |\bigcup_i L_i| + 1 \]

Due to symmetry along two axes (4 multiples):

\[ = 4 \cdot a + |\bigcup_i L_i| + 1, \text{ where } a \text{ is the number of lattice points along one axis, from values of } n, \text{ or } m, \text{ greater than zero.} \]

\( L_i \) describes only the lattice points in one quadrant, and there are four quadrants, so we must multiply the magnitude of \( L_i \) by four. The number of lattice points is then:

\[ = 4 \cdot a + 4|L_i| + 1 \]

We can rearrange:

\[ = 4 \cdot (a + |L_i|) + 1 \]

for \( k = (a + |L_i|) \),

\[ = 4 \cdot (k) + 1. \]