Parametric Curves and Surfaces

Parametric Curves in 3D

Maple can be of great help plotting and visualizing parametric curves and surfaces. Consider a parametric curve in the three-dimensional space given by

\[ x = x(t), \quad y = y(t), \quad z = z(t), \]

where the parameter \( t \) is changing in some interval \([a,b]\). The proper command for plotting parametric curves is spacecurve that we used before. The command is contained in the plots package. Let’s load the package and look at an example.

\[ \text{with(plots):} \]

Example 1. Consider an object moving in the xyz-space according to the equations

\[ x = t \cos(t), \quad y = t \sin(t), \quad z = t, \]

where \( x, y, z \) are measured in feet, \( t \) in minutes.

(a) Plot the object’s path for \( t \) between 0 and 20.
(b) Find the distance traveled by the object during that time.

Let’s define the coordinates \( x, y, z \) as expressions in terms of \( t \) and then plot the path.

\[ > \ x:=t*\cos(t)\; ; \; y:=t*\sin(t)\; ; \; z:=t; \]

\[ > \ x := t \cos(t) \]
\[ > \ y := t \sin(t) \]
\[ > \ z := t \]

Observe that the coordinates \( x, y, z \) under the spacecurve command (or formulas for those coordinates) are entered as a list, between square brackets.

\[ > \text{spacecurve([x,y,z],t=0..20,axes=normal,color=red,orientation=[54,64],labels=["x","y","z"]);} \]
Rotate the above graph to get a good sense of the motion. It’s a kind of circular motion around the vertical axis with the z-coordinate and the radius increasing.

To calculate the distance traveled we have to evaluate the integral of the speed between $t=0$ and $t=20$. We use the numerical integration right away as we suspect the resulting integral is going to be too difficult to evaluate otherwise.

\[
\begin{align*}
\int_0^{20} & \sqrt{(\cos(t) - t \sin(t))^2 + (\sin(t) + t \cos(t))^2 + 1} \, dt \\
& = 203.8429301
\end{align*}
\]

The object has traveled almost 204 feet during the first twenty minutes.

After you are finished with a particular parametric representation, it is very important to restore the variables $x$, $y$, $z$, to being just variables again. It is accomplished by using the following commands:

\[
\begin{align*}
&> \ x := 'x'; \ y := 'y'; \ z := 'z'; \\
&> x := x \\
&> y := y \\
&> z := z
\end{align*}
\]
Problem 1. An object is moving in the xyz-space along the trajectory defined by

\[ x = \ln(t + 1), \quad y = \cos(t), \quad z = \sin(t), \]

where \( x, y, z \) are measured in meters, \( t \) in minutes.

(a) Plot the object’s path for \( t \) between 0 and 25. Describe the motion in words.
(b) Calculate the distance traveled by the object during the 25 seconds.

Important Reminder! If you assign formulas to \( x, y, z \) do not forget to unassign them by using the commands \( x:='x' \), \( y:='y' \), \( z:='z' \).

**Parametric Surfaces in 3D**

A parametric surface in \( xyz \)-space is, in general, given by the set of equations

\[ x = x(s, t), \quad y = y(s, t), \quad z = z(s, t), \]

where \( s, t \) are parameters with specified ranges. Of course, the parameters may be denoted by letters other than \( s \) and \( t \). The basic syntax for plotting such surfaces uses the \texttt{plot3d} command and looks as in the following example.

**Example 2.** Plot the surface given by the parametric equations

\[ x = 4 \cos(\theta) + 2 \cos(\psi) \cos(\theta), \quad y = 4 \sin(\theta) + 2 \cos(\psi) \sin(\theta), \quad z = 2 \sin(\psi), \]

where the parameters \( \theta \) and \( \psi \) both change between 0 and 2 \( \pi \).

We don’t always have to define \( x, y, z \), as expressions in terms of the parameters before plotting a curve or a surface. We can enter the formulas for \( x, y, z \) directly under the \texttt{plot3d} command as shown below. Observe that the formulas for the \( x, y, z \) coordinates are entered as a \texttt{list} between square brackets. Unless you tell Maple otherwise, it assumes that the three expressions between the brackets are formulas for the Cartesian coordinates \( x, y, z \).

Greek letters are often used to denote parameters if the parameters have a geometric meaning of angles. Greek letters are entered using their full names. If you don’t like it, you can always relabel the parameters. For example, to \( s \) and \( t \).
Rotate the graph to get a good sense of the surface. The surface, called a torus, is a "doughnut" obtained by revolving a smaller circle, in our example of radius 2, along a bigger circle, in our example of radius 4. Try to see how the parameter $\theta$ is used to parametrize the big circle in the xy-plane, while $\psi$ is used to parametrize smaller circles with the centers at each point of the larger circle. The scaling=constrained option allows us to see clearly the circular elements of the surface.

**Problem 2.** Plot the surface given by the same parametric equations as in Example 2, but with the ranges for the parameters changed as follows:

(a) $\theta$ between 0 and $\pi$, $\psi$ between 0 and 2$\pi$. (b) $\theta$ between 0 and 2$\pi$ and $\psi$ between 0 and $\pi$.

Rotate the graph obtained in (b) to see clearly the effects of the change in the range on the surface. Are the resulting graphs what you expected?

**Example 3.** Find a parametric representation of the surface
We begin by rotating the surface to get a good sense of its shape. It seems that the surface is composed of circles with the center on the z-axis whose radius is changing as z is increasing from 0 to about 6. (Observe that the length of the unit on the x-axis and y-axis seems the same, so we do, indeed, have circles in planes parallel to the xy-plane.) For z=0, the radius is 1, then it decreases to 0 at about z=1.5, then increases again to 1. It seems that the radius is changing like \( \cos(z) \). Let’s try, then, the following parametrization

\[
x = \cos(s) \cos(\theta) \quad , \quad y = \cos(s) \sin(\theta) \quad , \quad z = s,
\]

for s and \( \theta \) between 0 and \( 2\pi \). The first two coordinates describe the circle with radius \( \cos(s) \) in the plane \( z = s \) parallel to the xy-plane. The radius changes like \( \cos(z) \). It should work. Let’s have Maple plot our parametric surface.

\[
\text{plot3d}([\cos(s)\cos(\theta),\cos(s)\sin(\theta),s],\theta=0..2*\pi,s=0..2*\pi,axes=framed,labels=["x","y","z"],orientation=[52,76]);
\]
It seems that our guess was correct!

**Problem 3.** Find a parametric representation of the following surfaces. Use Maple to check your answers.

(a)
Homework Problems
Your homework problems consist of Problems 1-3 above.

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