```
> restart;
```

Double Integrals

IMPORTANT: This worksheet depends on some programs we have written in Maple. You have to execute these first. Click on the + in the box below, then follow the directions you see at the beginning of the hidden section.

1 Programs for double integrals

This section contains Maple programs we will use. Place the cursor on the first red line below and hit return. When Maple is done processing, scroll back up to the line which reads "Programs for double integrals" close the section by clicking on the minus sign in the gray box to left.

```
>  with(plottools):  with(plots):
>  setwindow:=proc(xmin,xmax,ymin,ymax) global win;
>  if (nargs =0) then win:=’win’:  else win:=plot( 1000, x=xmin..xmax,
>  y=ymin..ymax):  fi; end:
>  plt:=proc(f,g,var,a,b)
>  if (evalf(a)<>evalf(b)) then plot([f,g,var=a..b]); fi; end:
>  polarpltr:=proc(f,g,var,a,b)
>  if (evalf(a)<>evalf(b)) then
>  polarplot([f,g,var=a..b],scaling=CONSTRAINED); fi; end:
>  polarplta:=proc(f,var,a,b)
>  if (evalf(a)<>evalf(b)) then
>  polarplot(f,var=a..b,scaling=CONSTRAINED); fi; end:
>  polarpltab:=proc(f,var,a,b)
>  if (evalf(a)<>evalf(b)) then
>  polarplot(f,var=a..b,scaling=CONSTRAINED,color=black); fi; end:
> dxdy command
>  dxdy:=proc(yrange,xrange) local a,b,f,g,h,Ls,i,PP1,PP2,PPtop,PPbot;
>  if (nargs <> 2 ) then
>  RETURN(print("Give TWO integration ranges.")); fi;
>  if (type(yrange,‘=‘) and (op(1,yrange)=y) and
>  type(op(2,yrange),range)
>  and type(xrange,‘=‘) and (op(1,xrange)=x) and
>  type(op(2,xrange),range)) then
>  a:=op(1,op(2,yrange));
>  b:=op(2,op(2,yrange));
>  f:=op(1,op(2,xrange));
>  g:=op(2,op(2,xrange));
>  else
>  RETURN(print("You need to give 2 integration ranges, separated
by a
>  comma:  y=a..b, x=ex1..ex2 so that the double integral would
have the
>  form", Int(Int(F(x,y),x=ex1..ex2),y=a..b))); fi;
>  if not(type(a,constant)) then
>  RETURN("Lower outside limit must be a number."); fi;
>  if not(type(b,constant)) then
>  RETURN("Upper outside limit must be a number."); fi;
>  if member(x,indets(f)) then
>  RETURN( "Lower inside limit must be in terms of y only.");  fi;
>  if member(x,indets(g)) then
>  RETURN( "Upper inside limit must be in terms of y only.");  fi;
>  h:=(b-a)/20;
>  print(Int(Int(F(x,y),x=f..g),y=a..b));
>  PP1:=plt(f,y,y,a,b);
>  PP2:=plt(g,y,y,a,b);
>  PPtop:=plt(x,b,x,subs(y=b,f),subs(y=b,g));
>  PPbot:=
>  plt(x,a,x,subs(y=a,f),subs(y=a,g)); Ls:=[seq(line([subs(y=a+i*h,f),
>  a+i*h],[subs(y=a+i*h,g),a+i*h]),i=1..19)];
>  if (win=’win’) then display(Ls,PP1,PP2,PPtop,PPbot); else
>  display(win,Ls,PP1,PP2,PPtop,PPbot); fi; end:
> dydx command
>  dydx:=proc(xrange,yrange) local a,b,f,g,h,Ls,i,PP1,PP2,PPleft,PPright;
>  if (nargs <> 2) then
>  RETURN(print("Give TWO integration ranges.")); fi;
>  if (type(yrange,‘=‘) and (op(1,yrange)=y) and
>  type(op(2,yrange),range)
>  and type(xrange,‘=‘) and (op(1,xrange)=x) and
>  type(op(2,xrange),range)) then
>  f:=op(1,op(2,yrange));
>  g:=op(2,op(2,yrange));
>  a:=op(1,op(2,xrange));
>  b:=op(2,op(2,xrange));
>  else
>  RETURN(print("You need to give 2 integration ranges, separated
by a
```

Scroll to top and close this section!

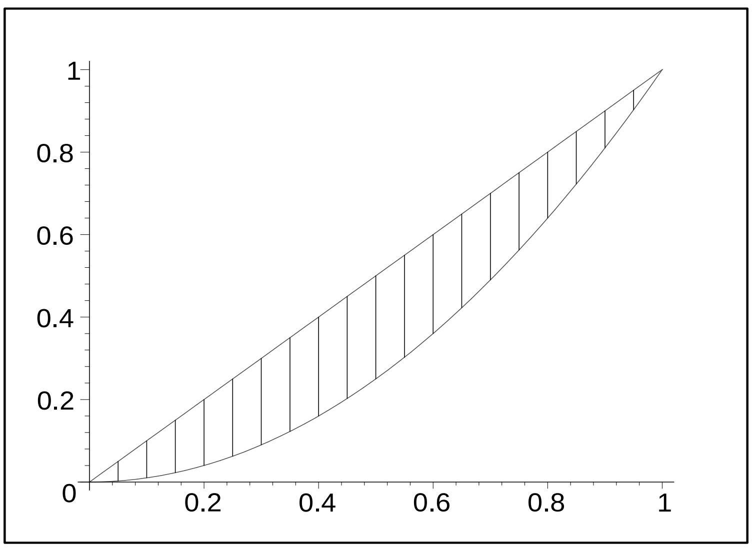
Part I, Rectangular Coordinates

Maple can calculate double integrals as iterated integrals, either numerically or by repeated use of the Fundamental Theorem of Calculus. The tricky part is providing the appropriate limits of integration, just as it is when doing such integrals by hand. For example, if R is the region, shown below, between the graphs of $y = x$ and $y = x^2$, and f$(x,\, y) = (x + y)^3$, we compute the integral of f over R as $\int_0^1 \int_{x^2}^{x} \mathrm{f}(x,\, y)\, dy\, dx$ as follows.

```
> int(int((x+y)^3,y = x^2 ..  x),x = 0 ..  1);
```

$$\frac{109}{504}$$

Region between y=x and $y = x^2$



If the function f(x,y) were less simple, for example if f$(x,\, y) = x \sin(x + y)$, we could compute the integral of f over the region R numerically as follows.

```
> evalf(Int(Int(x*sin(x+y),y = x^2 ..  x),x = 0 ..  1));
```

$$.06800130428$$

Observe that the Maple command for integrals has the integration range after the function, so that the "outer" integration range occurs last instead of first as it does when we write the double integral using integral signs.

In this worksheet, our focus will be on setting up double integrals, not evaluating them. To help you gain skill in doing this we have devised some new Maple commands, programmed in the subsection above, that will draw the region corresponding to the integration ranges you specify. For example, the region R appearing above was actually drawn by the new command dydx(x=0..1, y=xˆ2..x):, although we haven't displayed the command.

Notice that in the command dydx(x=0..1, y=xˆ2..x): the range for the outer integral (the "dx" integral in this case) is given first, as it is when writing integrals, although it appears last in the Maple command above.

Naturally, we programmed a similar command dxdy(y=...,x=...) for double integrals where the outer integral is with respect to y. If we reverse the order of integration in the integral above, we obtain

$\int_0^1 \int_{x^2}^{x} \mathrm{f}(x,\, y)\, dy\, dx = \int_0^1 \int_{y}^{\sqrt{y}} \mathrm{f}(x,\, y)\, dx\, dy$

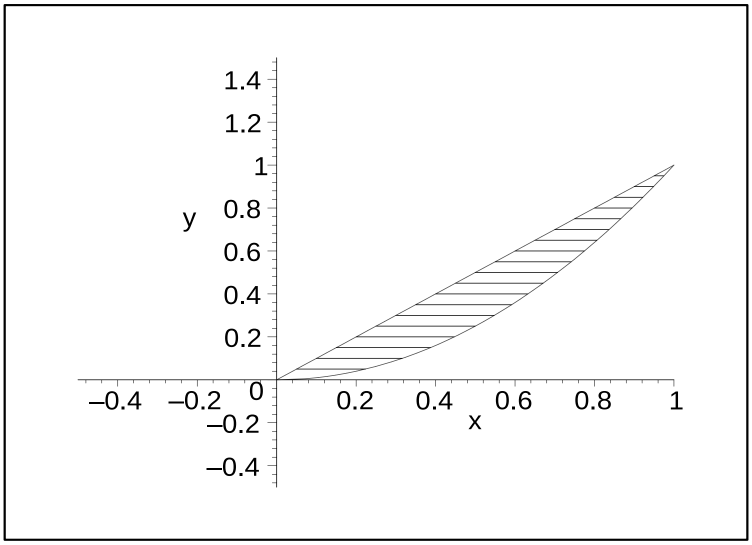
Hence, we expect the command dxdy(y=0..1, x=y..sqrt(y)) to produce the same region R. It does, indeed, as shown below.

When drawing regions Maple may choose a viewing window that isn't exactly what we want. Another new command, setwindow, will allow us to specify maximum and minimum values for x and y when using the dxdy and dydx commands. For example, if you want the viewing window be x=-5..5 and y=0..4, you can enter setwindow(-5,5,0,4): before the dxdy or dydx command, using the colon as shown. To return to letting Maple decide on the viewing window, just enter the command setwindow( ): using the empty parentheses as shown. You should reset the window this way for each new example or problem.

Let's actually use the dxdy command to see that we get the same region R that we discussed above. We will specify a view window with the setwindow command as well.

```
> setwindow(-.5,1,-.5,1.5):  dxdy(y=0..1, x=y..sqrt(y));
```

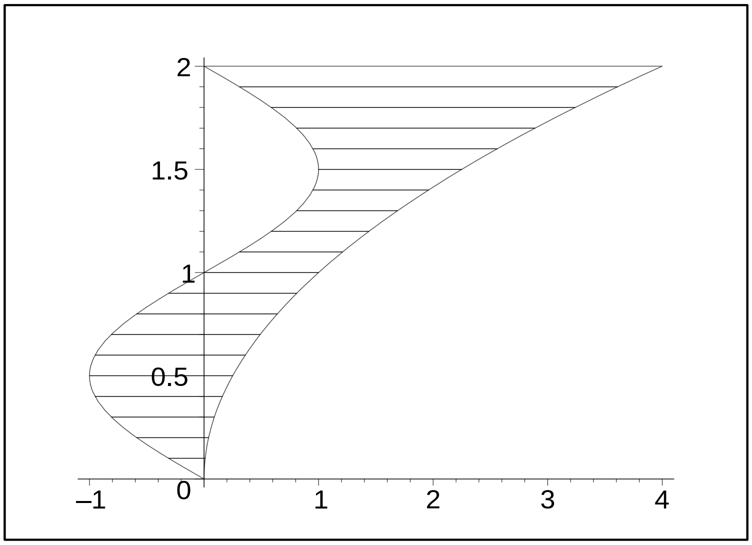
$$\int_0^1 \int_y^{\sqrt{y}} \mathrm{F}(x,\, y)\, dx\, dy$$



As you see, the dydx command will display the double integral in usual printed form and then draw the region. It will also try to catch some common errors and tell you about them, but it isn't foolproof. If you make some unanticipated error you will get one of Maple's usual error messages, or some unexpected result. You should also note that if you switch the upper and lower limits of integration on one of the integral signs, the region will remain the same. However, making this switch changes the sign of the integral.

Here's the game. We give you a region, and you have to use the dxdy or dydx commands to specify the limits of integration in a double integral that will integrate a function F(x,y) over the given region. For some regions, you will have to express the integral over the region as the sum of double integrals over simpler regions. We have reduced the size of graphic displays in the worksheet to save space, so expect those you generate using the dydx and other commands to be larger.

Example 1. Find integration limits for a double integral to integrate a function F(x,y) over the region shown below.



First of all, you should decide on the order of integration that will make the most sense for this region. Then see if you can describe the boundaries algebraically. Use the dxdy and/or dydx commands to try out your answers. Use setwindow( ): to have Maple decide on the view window.

You can get hints by clicking on the hidden sections below, but experiment yourself before peeking!!

```
> setwindow( ):
```

2 Hint 1

Use dxdy

3 Hint 2

The equation of the curve on the left has the form x = A*sin(B*y). What are A and B?

4 Hint 3

The curve on the right has equation $y = \sqrt{x}$

5 Solution

setwindow():dxdy(y=0..2,x=-sin(Pi*y)..y^2);

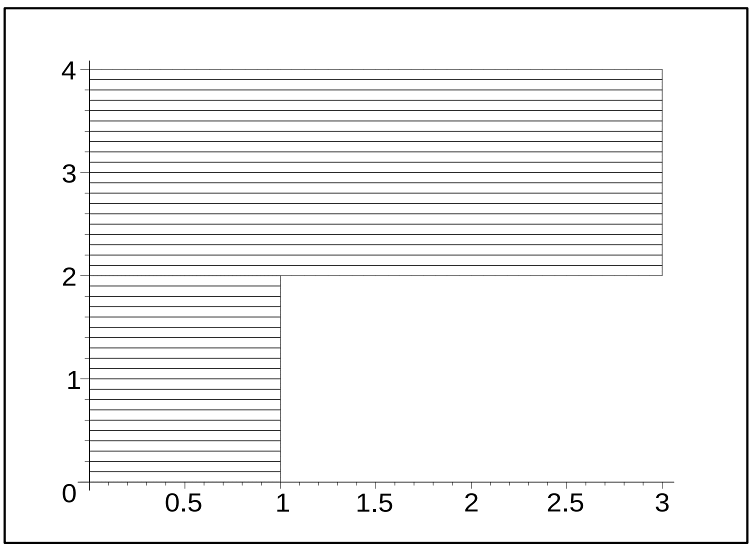
This command corresponds the the double integral

$\int_0^2 \int_{-\sin(\pi\, y)}^{y^2} \mathrm{f}(x,\, y)\, dx\, dx$

Example 2. To integrate over the L-shaped region below we might break the region into two rectangular regions and add the integrals over these rectangular regions. For example, the integral of F(x,y) over the region shown below can be written $\int_0^2 \int_0^1 \mathrm{F}(x,\, y)\, dx\, dy + \int_2^4 \int_0^3 \mathrm{F}(x,\, y)\, dx\, dy$ and we can draw the region by the command display(dxdy(y=0..2,x=0..1), dxdy(y=2..4,x=0..3));

Note how the we use the display command to combine the results of the two dxdy commands.

How can you write the integral over the region as the sum of two "dydx" (rather than "dxdy") double integrals? How could you use the display command with two dydx commands to draw the region?

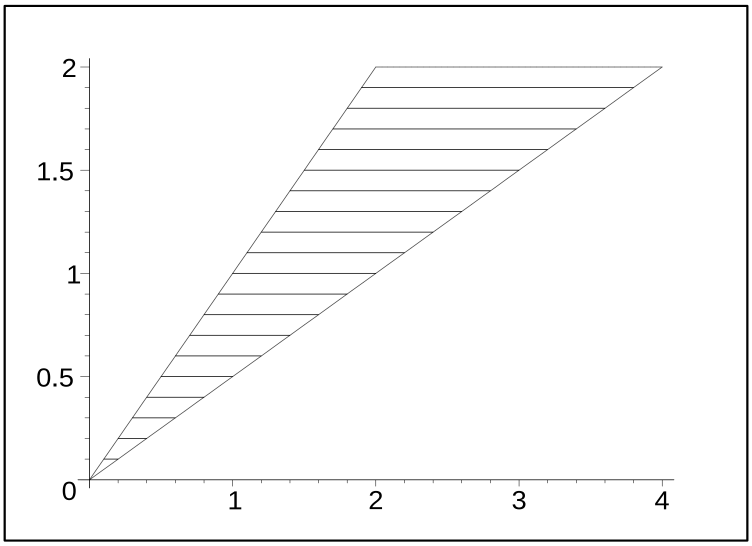


Homework Problems, Part I

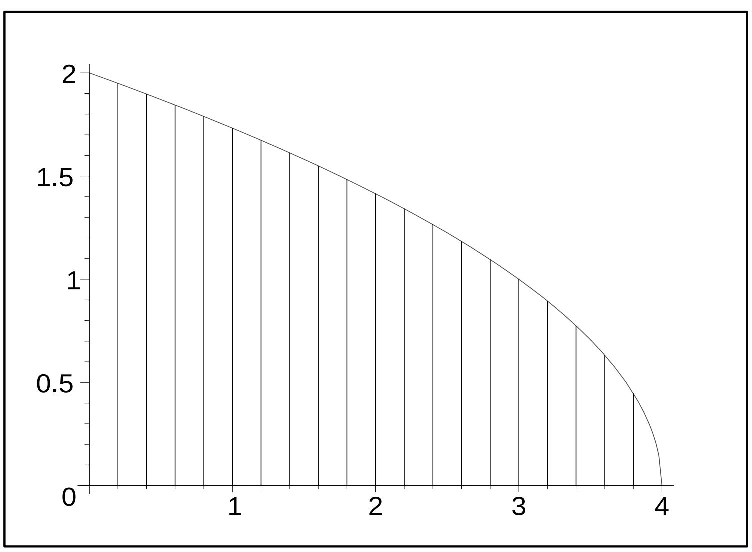
For each of the following regions and for an arbitrary function F(x,y), write a double integral of "dxdy" or "dydx" type (or both if specified) of F(x,y) over the region. Use the dxdy and dydx commands to experiment, check your work and produce your final answer. *Remember that if you use the setwindow command, the view window will remain in effect until you release it by using the command: setwindow( ):.*

Problem 1. Give both "dxdy" and "dydx" integrals of F(x,y) over the region:

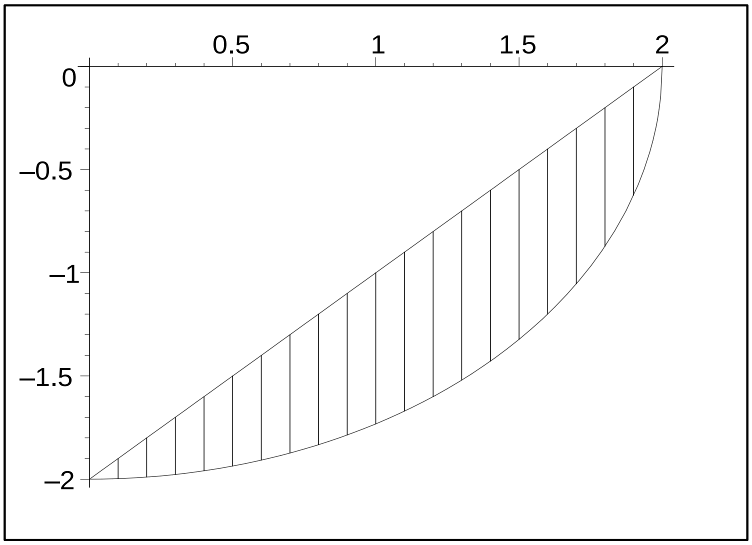
.



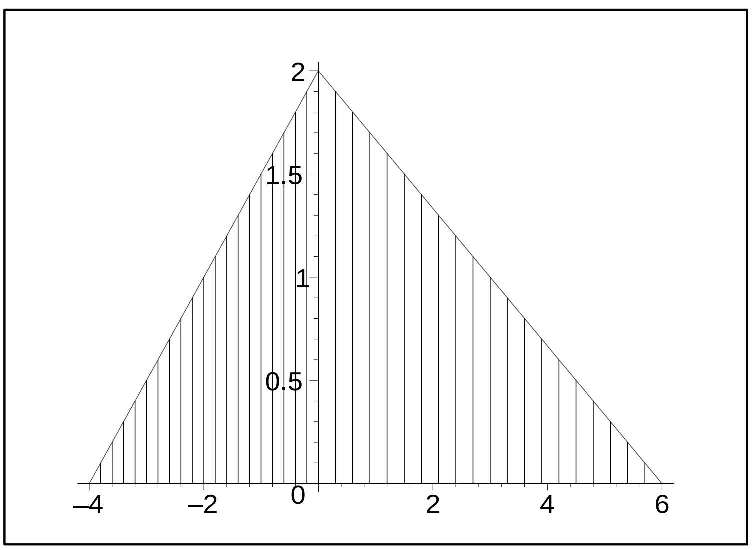
Problem 2. The curved boundary is a parabola. Give both "dxdy" and "dydx" integrals of F(x,y) over the region:



Problem 3. The curved part is an arc of a circle. Give both "dxdy" and "dydx" integrals.



Problem 4. Give both "dxdy" and "dydx" integrals.



Part II, Polar Coordinates

In the subsection at the beginning of this worksheet, we have programmed two other commands, drdtheta, and dthetadr, that draw regions corresponding to double integrals in polar coordinates. The setwindow command has no effect on these polar coordinate commands. You will find it helpful to know a little more about plotting certain curves in polar coordinates, especially in example 2 below. Click the + below for more on polar plotting.

6 Polar plotting in brief

Execute the following command first.

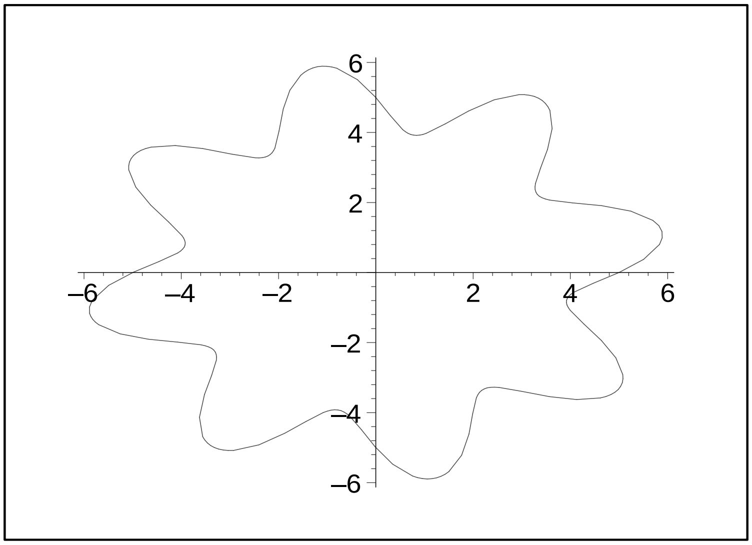
```
> with(plots):
```

Curves are frequently described in polar coordinates by equations like
$r = \mathrm{f}(\theta)$
with a range of values for θ. The corresponding curve is then the set of points in the plane that have a pair of polar coordinates (r, θ) satisfying the equation. We agree that if r is negative, we identify (r, θ) with the point we get by reflecting (-r, θ) through the origin, so for example, the point with rectangular coordinates (1,1) can be represented in polar coordinates as ($\sqrt{2}$, $\frac{\pi}{4}$) or as (- $\sqrt{2}$, $\frac{5\,\pi}{4}$).

The easiest example is provided by the equation $r = c$ where c is a positive constant, i.e. where the function f(θ) is constant. The corresponding curve is then obviously a circle of radius c. We can plot r= f(θ) by plotting the point with polar coordinates (f(θ), θ) for each θ in the given range. Thus we can think of varying the angle θ and, for each angle, plotting the point lying at distance f(θ) from the origin along the ray making angle θ with the positive x-axis. If f(θ) is negative we go backward along the ray. Maple uses the command polarplot to do this. Study the following examples.

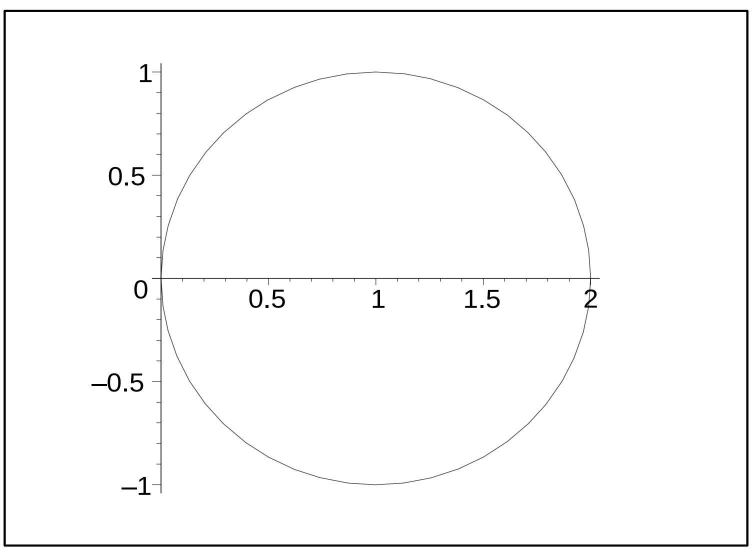
To plot $r = 5 + \sin(8\,\theta)$ for θ from 0 to 2 π:

```
> polarplot(5+sin(8*theta), theta=0..2*Pi);
```



Here is the plot of $r = 2\cos(\theta)$ for θ from 0 to π. Note that in this example r is negative for θ between $\frac{\pi}{2}$ and π, and these values of θ correspond to the bottom half of the circle you see.

```
> polarplot(2*cos(theta), theta=0..Pi,scaling = constrained);
```



To see that $r = 2\cos(\theta)$ really does give the circle with center at (1,0) and radius 1, as it appears to, we can proceed as follows. Recall that the equation of this circle in rectangular coordinates is $(x-1)^2 + y^2 - 1 = 0$, and that polar and rectangular coordinates are related by the equations $x = r\cos(\theta)$, and $y = r\sin(\theta)$. Substitute the last two equations into the first equation and simplify. Let's have Maple do the algebra.

```
> subs(x=r*cos(theta),y=r*sin(theta), (x-1)^2+y^2-1=0); expand(%);
> simplify(%/r);
```

$$(r\cos(\theta) - 1)^2 + r^2\sin(\theta)^2 - 1 = 0$$

$$r^2\cos(\theta)^2 - 2\,r\cos(\theta) + r^2\sin(\theta)^2 = 0$$

$$-2\cos(\theta) + r = 0$$

The polar equation is on the last line. What do you think $r = 2\sin(\theta)$ looks like? How about $r = -9\cos(\theta)$?

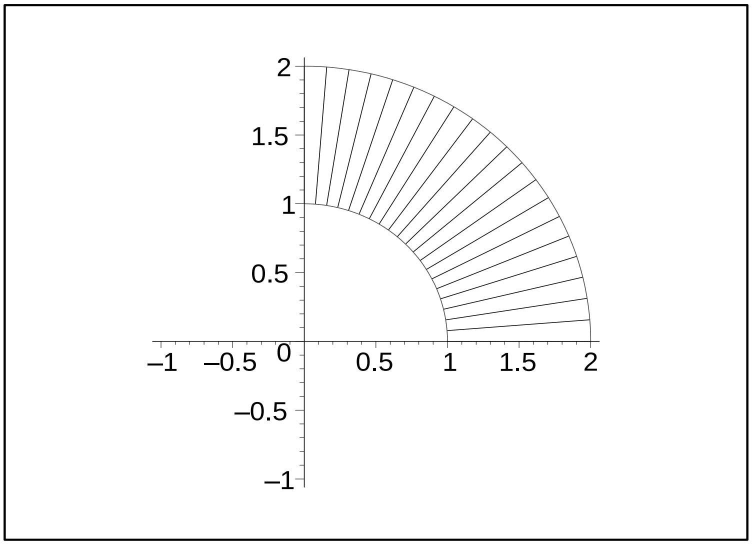
The polar equations lines through the origin have simple equations, name, $\theta = c$ where c is a constant. For other lines we can find the polar equation by first writing an equation for the line in rectangular coordinates and then substituting $x = r\cos(\theta)$, and $y = r\sin(\theta)$. For example, the vertical line x=5 has polar equation $r\cos(\theta) = 5$ or $r = \frac{5}{\cos(\theta)}$. What do you expect the command polarplot(3/sin(theta), theta=Pi/4..Pi/2); to produce?

END OF HIDDEN SECTION. Scroll up to close.

Example 3.

```
> drdtheta(theta= 0..Pi/2, r=1..2);
```

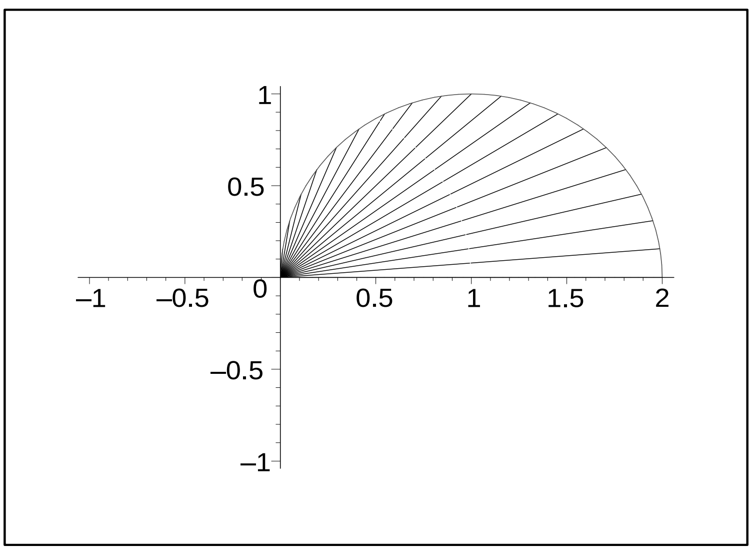
$$\int_0^{1/2\,\pi}\int_1^2 \mathrm{F}(r,\,\theta)\,r\,dr\,d\theta$$



Example 4.

```
> drdtheta(theta=0..Pi/2, r =0..  2*cos(theta));
```

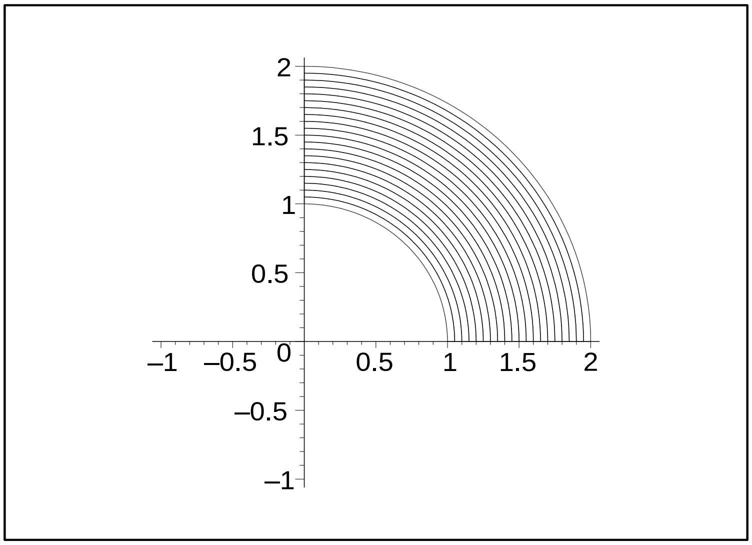
$$\int_0^{1/2\,\pi}\int_0^{2\cos(\theta)} \mathrm{F}(r,\,\theta)\,r\,dr\,d\theta$$



Example 5. We can integrate over the region in Example 1 with a "d θdr" integral:

```
> dthetadr(r=1..2, theta=0..Pi/2);
```

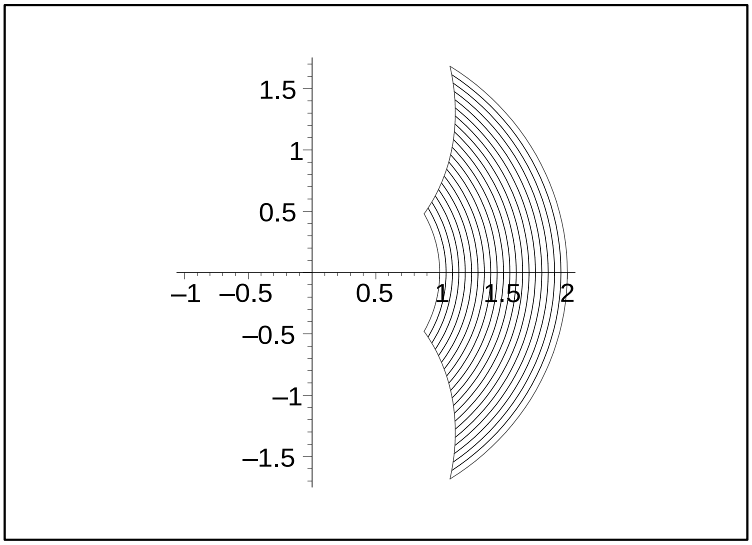
$$\int_1^2 \int_0^{1/2\,\pi} \mathrm{F}(r,\,\theta)\, r\, d\theta\, dr$$



Example 6. You might have to think a bit about why the region looks the way it does here:

```
> dthetadr(r=1..2, theta=-r/2..r/2);
```

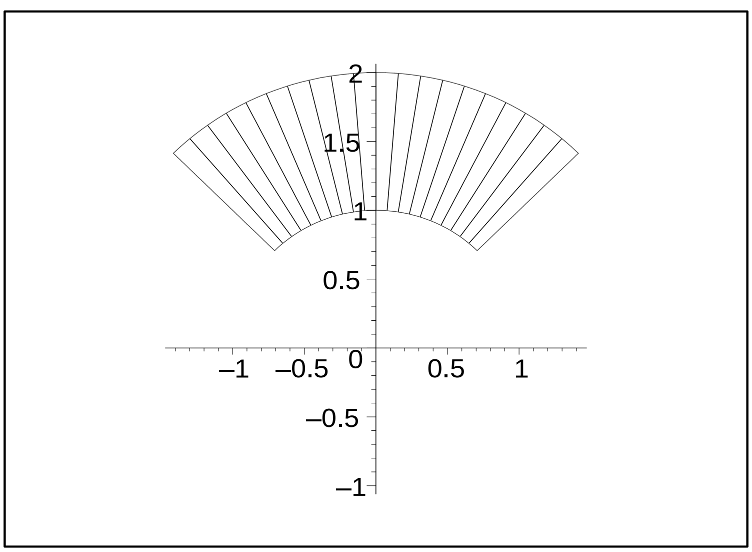
$$\int_1^2 \int_{-1/2\,r}^{1/2\,r} \mathrm{F}(r,\,\theta)\, r\, d\theta\, dr$$



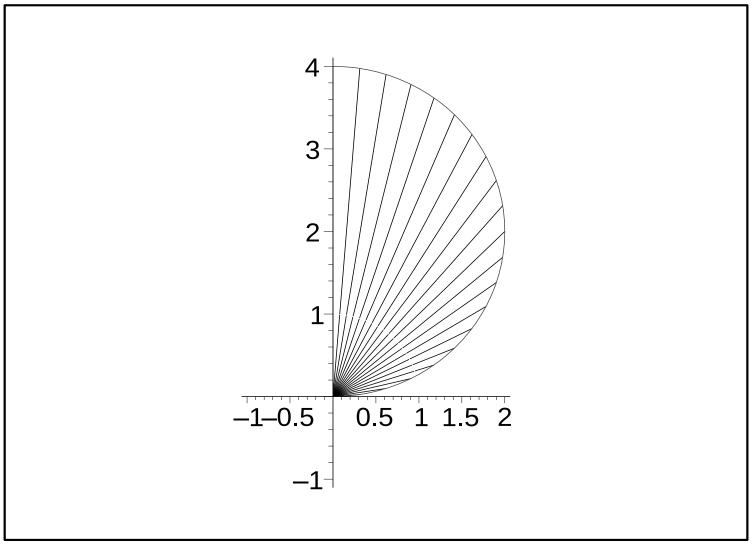
Homework Problems, Part II

For each of the following regions, write a double integral of "drd θ" or "d θdr" type, or both if specified, for an function F(r, θ). Use thedrdtheta and dthetadr commands to experiment and check your work.

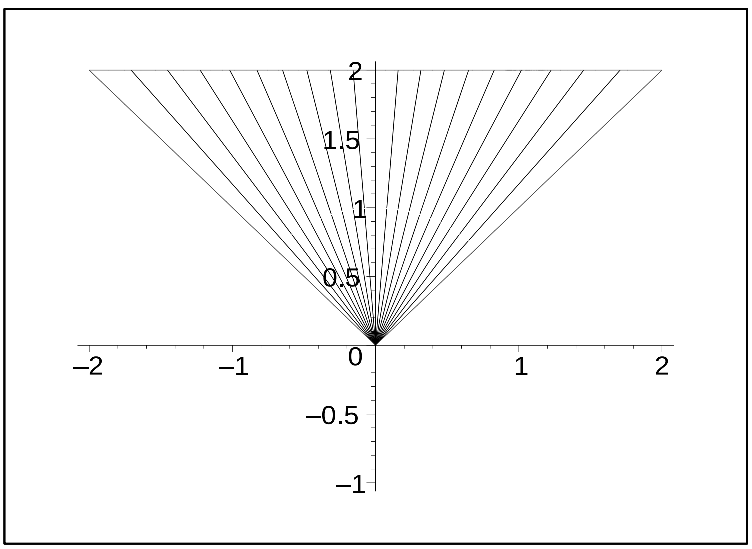
Problem 5. Do both ways.



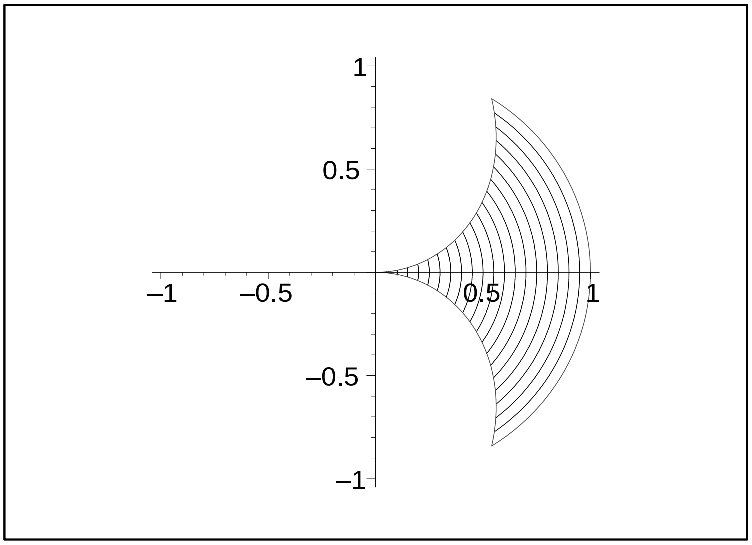
Problem 6.



Problem 7.



Problem 8. Do some experiments!


1
0.5
–1
–0.5
0.5
1
–0.5
–1