The Fundamental Theorem of Calculus

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OK!

1 The Fundamental Theorem of Calculus

In this handout we will review the Fundamental Theorem of Calculus (**FTC**), which typically is discussed in any first calculus course. The reason for including **FTC** in MTH 362 course is that we will need it in our discussions of probability and density functions, and also to better prepare you to study applications that arise in electrical engineering courses.

Recall that a definite integral of a continuous function $f(t)$ on an interval (a, b) is defined as a limit of Riemann sums:

$$\int_a^b f(t)dt = \lim_{\Delta t \to 0} \sum_{\ell=1}^{n} f(t_\ell)\Delta t$$

So, in principle, it is always possible to calculate $\int_a^b f(t)dt$ approximately (by using a computer program, for example).

The form of **FTC** that probably you are most familiar with is given next. It is not the most general version or the most polished [1] , but for our purposes is adequate.

Theorem 1.1 (FTC Version A) *Let f and F be two functions such that*

a. f and F are continuous on the iterval closed interval $[a, b]$.

b. F is an antiderivative of f on the open interval (a, b), that is, $F'(t) = f(t)$ for $t \in (a, b)$.

Then,

$$\int_a^b f(t)dt = F(b) - F(a) \tag{1}$$

In words, the definite integral of the rate of change is equal to the total change.

Before we state a second version of **FTC** , we must mention an important fact. If f is any continuous function on $[a, b]$, it is always possible to define a new function F on $[a, b]$ as follows:

$$F(x) = \int_a^x f(t)dt, \qquad a \leq x \leq b$$

Clearly, $F(a) = 0$. Also, one can prove that the function F is continuous.

Theorem 1.2 (FTC Version B) *Let f be continuous function on $[a, b]$, and let F be defined by*

$$F(x) := \int_a^x f(t)dt, \qquad a \leq x \leq b$$

Then F is an antiderivative of f on (a, b), that is,

$$F'(x) = f(x) \quad \textit{for} \quad x \in (a, b)$$

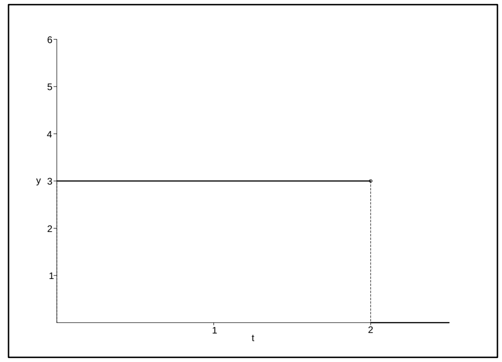
Moreover, F is the only antiderivative of f that is continuous on $[a, b]$ and that satisfies $F(a) = 0$.

[1] The hypotheses in the theorem may be stated in various slightly different ways. The trade-off is: a simple statement but not so general, versus a general statement but more technical (complicated)

2 Graphical Examples

2.1 Example

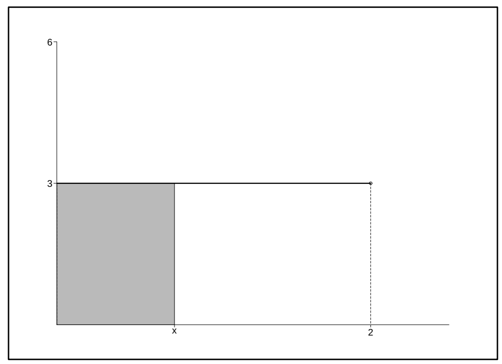
A constant function on the interval $[0, 2]$.



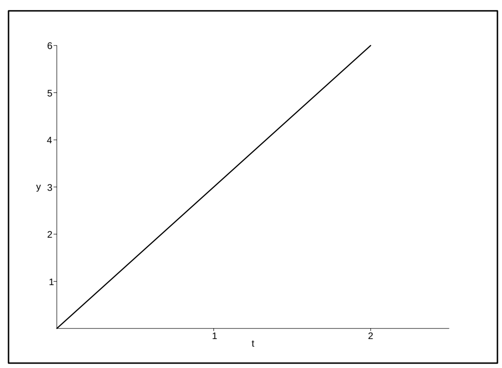
As x moves from 0 to 2, the quantity

$$F(x) = \int_0^x f(t)dt$$

(which represents the area under the curve between 0 and x) increases. Clearly such area is given by the formula $3x$.



The plot of the function $F(x) = \int_0^x f(t)dt$



2.2 Example

Consider a function $f(x)$ that is piecewise constant as shown in the figure. One can see that,

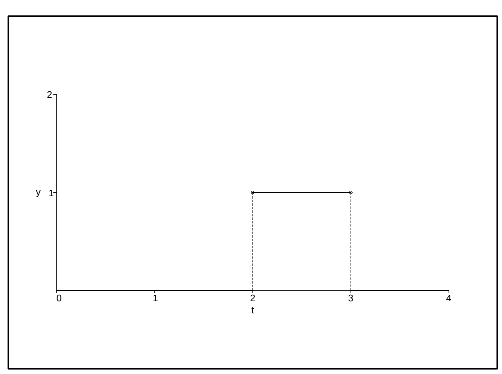
- when x is between 0 and 2,

$$F(x) = \int_0^x f(t)dt \quad \text{is equal to } 0$$

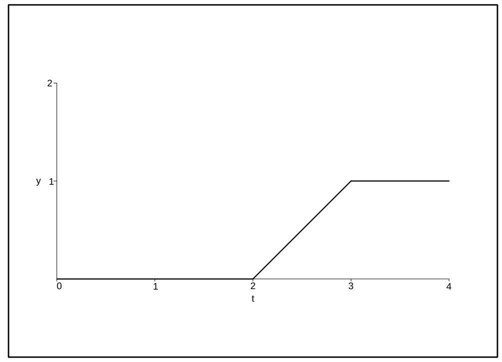
- when x is between 2 and 3,

$$F(x) = \int_0^x f(t)dt \quad \text{increases linearly to } 1$$

- when x is larger than 3 and increases, $F(x) = \int_0^x f(t)dt$ remains constant at 1

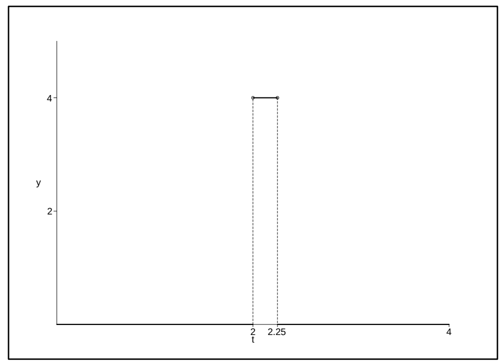


The plot of the function $F(x) = \int_0^x f(t)dt$

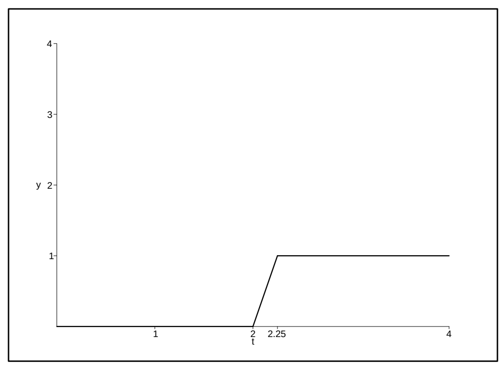


2.3 Example

We now look at a "rescaled" version of the previous example. It might represent an "impulse" near 2.

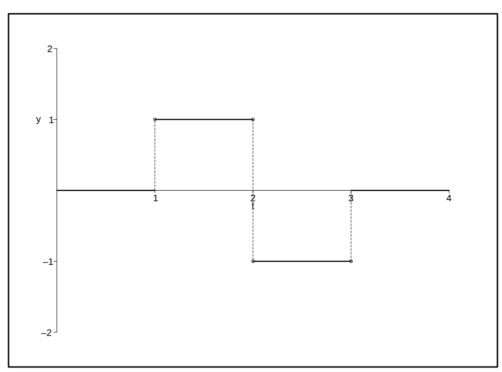


Note that the constant value of $F(x)$ for $x > 2.25$ is equal to the area of the tall rectangle in the graph of f.

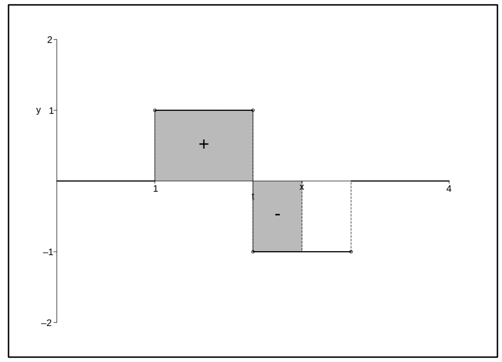


2.4 Example

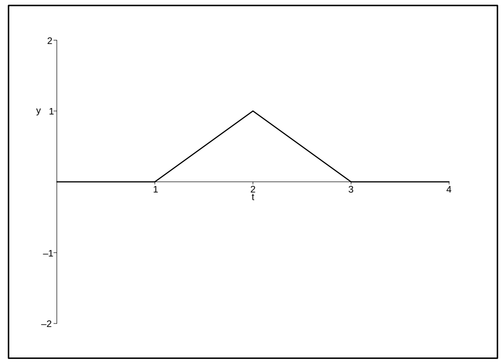
Now we look at a function f which has both positive and negative values, as given by the graph in the figure.



As x goes from 1 to 2, $F(x) = \int_1^x f(t)dt$ increases linearly from 0 to 1, but as x moves from 1 to 2 we *subtract* the area between the graph of f and the x-axis between 2 and x.

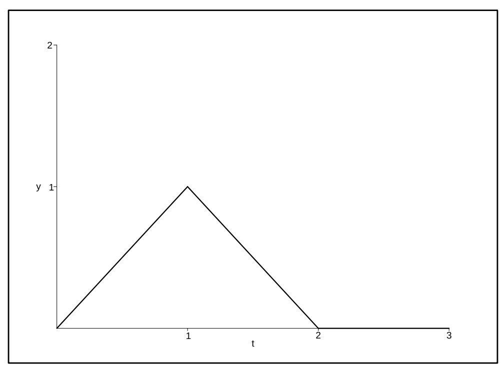


The plot of the function $F(x) = \int_0^x f(t)dt$



2.5 Example

Suppose the graph of f is as in the figure.



As x goes from 0 to 1, the formula for the area of a triangle shows that

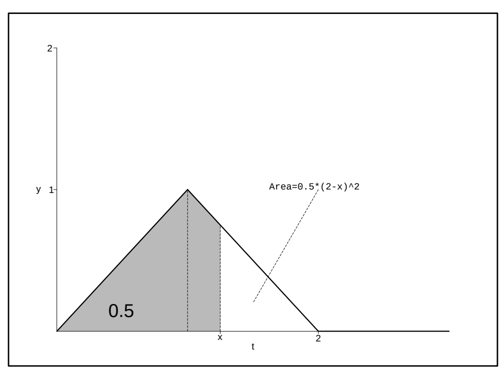
$$\int_0^x f = F(x) = x^2/2$$

so $F(1) = 1/2$. Now if x is between 1 and 2 the area formula gives

$$F(x) = 1/2 + (1/2 - (2-x)^2/2)$$

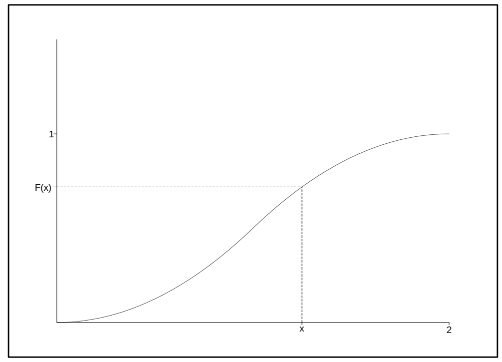
We write this as follows:

$$F(x) = \begin{cases} x^2/2 & \text{if } 0 \le x \le 1 \\ 1 - (2-x)^2/2 & \text{if } 1 \le x \le 2 \end{cases} \quad (1)$$



We could have found the formula for $F(x)$ when x is between 1 and 2 by noting that the graph of f coincides with the graph of the line $y = -x + 2$ when x is between 1 and 2. Then for such x,

$$F(x) = \frac{1}{2} + \int_1^x (-t+2)\,dt = \frac{1}{2} + [-(-t+2)^2/2]_1^x = 1 - (2-x)^2/2$$



3 An algebraic example

Consider the function $f(t) = te^{-t^2}$. The function $F(x) = \int_0^x f(t)dt$ may be calculated to obtain a formula that has no integrals in it. Indeed, it is easy to see that

$$F(x) = \frac{-1}{2}e^{-t^2}\Big|_0^x = \frac{-1}{2}(e^{-x^2} - 1)$$

Let us now consider the function $f(t) = e^{-t^2}$. In this case it is not possible to obtain a formula for $F(t)$ without integrals in it. However, one may "handle" this numerically with the help of a computer algebra system such as Maple or Mathematica. This we shall do in the next section.

4 An example with Maple

The function $f(t) = e^{-t^2}$ was introduced in the previous section. In Maple we may type

```
> f := x -> exp(-t^2);
```

$$f := t \;\rightarrow\; e^{-t^2}$$

We now compute F as follows:

```
> F := x -> evalf(Int(f(t),t=0..x));
```

$$F := x \;\rightarrow \text{evalf}\left(\int_0^x f(t)dt\right)$$

Note that by using the evalf command along with the inert form of the integral command (Int vs. int), we are telling Maple to compute values of F(x) by using numerical integration. The plots of $f(t)$ and $F(x)$ may be generated in Maple in the usual way.

5 A Physical Example

With suitable units, the electrical charge, q, at a circuit element is a function of time t: $q = q(t)$. The current, i, is defined to be the rate of change of current with respect to time, i.e., $i = dq/dt$, so that $i = i(t)$ is a function of time also.

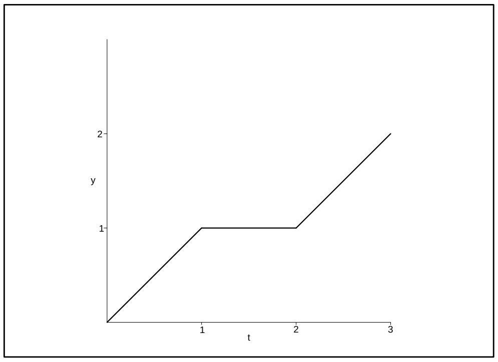
If we know the values of $i(t)$ for values of t in the time interval $[0, 10]$ say, then we can determine, for any T between 0 and 10, the value of $q(T)$, which will give the total charge that has flowed into the circuit element between times 0 and T:

$$q(T) = \int_0^T i(t)\,dt.$$

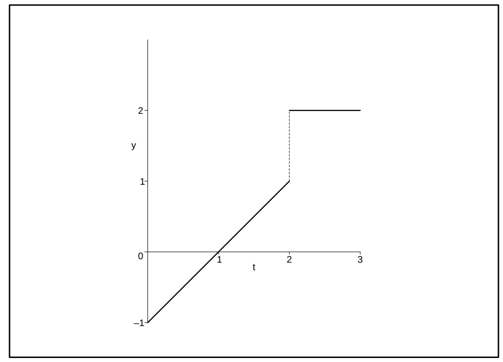
6 Problems

1. For each of the graphs given below, sketch a graph of the corresponding antiderivative $F(x) = \int_0^x f(t)dt$.
2. For problems A, B, and C, determine a formula for $F(x)$ (similar to what you saw in Example 2.5.

(A)



(B)



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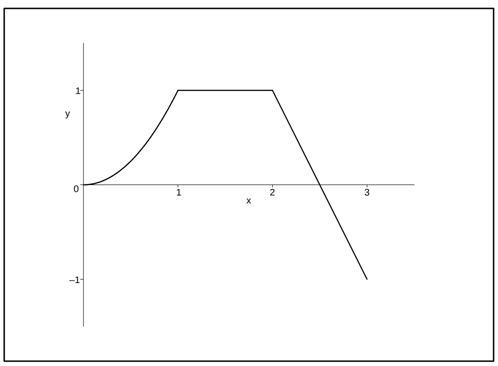

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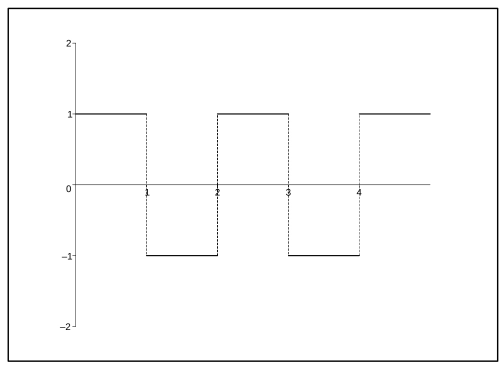

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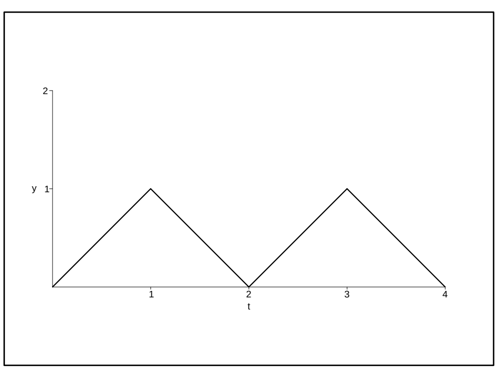

By O. Merino 9/2000. Previous version 1/2000 by L. Pakula.