The Global Character of Solutions for a System of Difference Equations

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Problem (1):

Investigate the global character of solutions of the system of difference equations

\[
\begin{align*}
    x_{n+1} &= \frac{\alpha_1 + \beta_1 x_n + \gamma_1 y_n}{A_1 + B_1 x_n + C_1 y_n}, \quad \text{for } n=0, 1, 2, \ldots \\
    y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n}
\end{align*}
\]

where the parameters and initial terms are non-negative numbers such that the denominators are never equal to zero.
Problem (1):

Investigate the global character of solutions of the system of difference equations

\[ x_{n+1} = \frac{\alpha_1 + \beta_1 x_n + \gamma_1 y_n}{A_1 + B_1 x_n + C_1 y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n}, \quad \text{for } n=0,1,2,... \]

where the parameters and initial terms are non-negative numbers such that the denominators are never equal to zero.

In 2008, professors Camouzis, Kulenovic, Ladas, and Merino [CKLM] introduced a numbering system which divided the problem into 2,401 special cases.
Equations for $x_{n+1}$:

1. $x_{n+1} = \frac{\alpha_1}{A_1}$
2. $x_{n+1} = \frac{\alpha_1}{B_1 x_n}$
3. $x_{n+1} = \frac{\alpha_1}{C_1 y_n}$

\vdots

48. $x_{n+1} = \frac{\alpha_1 + \beta_1 x_n + \gamma_1 y_n}{B_1 x_n + C_1 y_n}$

49. $x_{n+1} = \frac{\alpha_1 + \beta_1 x_n + \gamma_1 y_n}{A_1 + B_1 x_n + C_1 y_n}$

Equations for $y_{n+1}$:

1. $y_{n+1} = \frac{\alpha_2}{A_2}$
2. $y_{n+1} = \frac{\alpha_2}{C_2 y_n}$
3. $y_{n+1} = \frac{\alpha_2}{B_2 x_n}$

\vdots

48. $y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{B_2 x_n + C_2 y_n}$

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\[
\begin{align*}
x_{n+1} &= \frac{\alpha_1 + \beta_1 x_n + \gamma_1 y_n}{A_1 + B_1 x_n + C_1 y_n} \\
y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n}
\end{align*}
\], \(n=0,1,2,\ldots\)

System \#(3,48):
\[
\begin{align*}
x_{n+1} &= \frac{\alpha_1}{C_1 y_n} \\
y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{B_2 x_n + C_2 y_n}
\end{align*}
\], \(n=0,1,2,\ldots\)
Some Applications of Problem (1):
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Beverton-Holt Equation [BH]

\[ x_{n+1} = \frac{\beta_1 x_n}{A_1 + B_1 x_n}, \text{ for } n=0,1,2,... \]

This is Equation #13
Some Applications of Problem (1):

Beverton-Holt Equation [BH]

\[ x_{n+1} = \frac{\beta x_n}{A_1 + B_1 x_n}, \text{ for } n=0,1,2,... \]  \hspace{1cm} \text{This is Equation #13}

Leslie-Gower Stage Structured Model [CHR]

\[
\begin{align*}
J_{t+1} &= b \cdot \frac{1}{1 + d \cdot A_t} \cdot A_t, \text{ for } t=0,1,2,... \hspace{1cm} \text{This is System #(17,17)} \\
A_{t+1} &= s \cdot \frac{1}{1 + c \cdot J_t} \cdot J_t
\end{align*}
\]

Leslie-Gower Competitive Model [CLCH]

\[
\begin{align*}
x_{t+1} &= b_1 \cdot \frac{1}{1 + c_{11} x_t + c_{12} y_t} \cdot x_t, \text{ for } t=0,1,2,... \hspace{1cm} \text{This is System #(38,38)} \\
y_{t+1} &= b_2 \cdot \frac{1}{1 + c_{21} x_t + c_{22} y_t} \cdot y_t
\end{align*}
\]
Some Applications of Problem (1):

Pielou’s Equation [P]  
\[ x_{n+1} = \frac{\beta x_n}{1 + x_{n-1}} \]

can be transformed into 

System #(14,7):  
\[ x_{n+1} = \frac{\beta_1 x_n}{A_1 + C_1 y_n} \]
\[ y_{n+1} = \beta_2 x_n \]

, for \( n=0,1,2,... \)
Some Applications of Problem (1):

Pielou’s Equation [P] \( x_{n+1} = \frac{\beta x_n}{1+x_{n-1}} \) can be transformed into

System #(14,7): \[
\begin{aligned}
x_{n+1} &= \frac{\beta_1 x_n}{A_1 + C_1 y_n}, & \text{for } n=0,1,2,... \\
y_{n+1} &= \beta_2 x_n
\end{aligned}
\]

May’s Host Parasitoid Model [M] \[
\begin{aligned}
x_{n+1} &= \frac{\alpha x_n}{1+y_n} \\
y_{n+1} &= \frac{x_n y_n}{1+y_n}
\end{aligned}
\]

into System #(6,25): \[
\begin{aligned}
x_{n+1} &= \frac{\beta_1 x_n}{C_1 y_n}, & \text{for } n=0,1,2,... \\
y_{n+1} &= \beta_2 x_n + \gamma_2 y_n
\end{aligned}
\]
Problem (1):

\[
\begin{align*}
x_{n+1} &= \frac{\alpha_1 + \beta_1 x_n + \gamma_1 y_n}{A_1 + B_1 x_n + C_1 y_n}, \\
y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n},
\end{align*}
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for \( n = 0, 1, 2, \ldots \)

Analysis of a system of difference equations:
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for \( n=0,1,2,... \)

Analysis of a system of difference equations:

(1) Determine the boundedness character of the solutions
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x_{n+1} = \frac{\alpha_1 + \beta_1 x_n + \gamma_1 y_n}{A_1 + B_1 x_n + C_1 y_n}, \text{ for } n=0,1,2,... \\
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Analysis of a system of difference equations:

(1) Determine the boundedness character of the solutions

(2) Perform a local linearized stability analysis of the equilibrium points and periodic points
Problem (1):

\[
x_{n+1} = \frac{\alpha_1 + \beta_1 x_n + \gamma_1 y_n}{A_1 + B_1 x_n + C_1 y_n} \\
y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n}
\]

, for \( n = 0, 1, 2, \ldots \)

Analysis of a system of difference equations:

(1) Determine the boundedness character of the solutions

(2) Perform a local linearized stability analysis of the equilibrium points and periodic points

(3) Determine the global behavior of the solutions
“The art of doing mathematics consists in finding that special case which contains all the germs of generality.”

- David Hilbert
Extensions of Problem (1):
Extensions of Problem (1):

(i) Replace coefficients with periodic sequences [CL] [KM]

\[
\begin{align*}
x_{n+1} &= \frac{\alpha_n + \beta_n x_n + \gamma_n y_n}{A_n + B_n x_n + C_n y_n}, \text{ for } n=0,1,2,\ldots \\
y_{n+1} &= \frac{\delta_n + \varepsilon_n x_n + \phi_n y_n}{E_n + D_n x_n + F_n y_n}
\end{align*}
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Extensions of Problem (1):

(i) Replace coefficients with periodic sequences    [CL] [KM]

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\]
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\]

, for \( n=0,1,2,... \)

(ii) Use higher-order equations    [LP]

\[
x_{n+1} = \frac{\alpha + \sum_{j=0}^{k} \beta_j x_{n-j} + \sum_{j=0}^{k} \gamma_j y_{n-j}}{A + \sum_{j=0}^{k} B_j x_{n-j} + \sum_{j=0}^{k} C_j y_{n-j}}
\]
\[
y_{n+1} = \frac{\delta + \sum_{j=0}^{k} \varepsilon_j x_{n-j} + \sum_{j=0}^{k} \phi_j y_{n-j}}{D + \sum_{j=0}^{k} E_j x_{n-j} + \sum_{j=0}^{k} F_j y_{n-j}}
\]

, for \( n=0,1,2,... \)
Extensions of Problem (1):

(iii) Systems of 3 equations

\[
\begin{align*}
    x_{n+1} &= \frac{\alpha_1 + \beta_1 x_n + \gamma_1 y_n + \delta_1 z_n}{A_1 + B_1 x_n + C_1 y_n + D_1 z_n} \\
y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n + \delta_2 z_n}{A_2 + B_2 x_n + C_2 y_n + D_2 z_n} \\
z_{n+1} &= \frac{\alpha_3 + \beta_3 x_n + \gamma_3 y_n + \delta_3 z_n}{A_3 + B_3 x_n + C_3 y_n + D_3 z_n}
\end{align*}
\]

, for \(n=0,1,2,...\)

There are 2,401 systems of equations contained in Problem (1). Using a similar numbering system, there are 11,390,625 systems of equations contained in this extension of Problem (1).
Boundedness Results for Problem (1):
Boundedness Results for Problem (1):

* The boundedness character of the solutions has been established for every system on the list, except for System #(6,25).

[CDL] [CKLL] [LP] [BCLL] [CLW] [KM]

System #(6,25): 

\[
\begin{align*}
    x_{n+1} &= \frac{\beta_1 x_n}{C_1 y_n} \\
    y_{n+1} &= \beta_2 x_n + \gamma_2 y_n
\end{align*}
\]

, for \( n=0,1,2,... \)
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\begin{align*}
    x_{n+1} &= \frac{\beta_1 x_n}{C_1 y_n}, \quad \text{for } n=0,1,2,\ldots \\
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\end{align*}
\]

* The boundedness character of May’s Host Parasitoid Model was conjectured by Dr. Ladas in 1995. That conjecture can be reformulated into the conjecture that System #(6,25) is (B*,U).
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* The boundedness character of the solutions has been established for every system on the list, except for System #(6,25).

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System #(6,25): \[ x_{n+1} = \frac{\beta_1 x_n}{C_1 y_n}, \text{ for } n=0,1,2,... \]
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* The boundedness character of May’s Host Parasitoid Model was conjectured by Dr. Ladas in 1995. That conjecture can be reformulated into the conjecture that System #(6,25) is (B*,U).

* If this conjecture is true, then for the 49 equations on the list, there are only 15 distinct patterns of boundedness.
Let the $x_{n+1}$ equation be either equation #8 or #23 on the list.

Let the $y_{n+1}$ equation be any of the 49 equations on the list.

Let \( \{x_n, y_n\} \) be a solution of the system of equations

\[
\begin{align*}
  x_{n+1} &= \frac{\alpha_1 + y_n}{x_n} \\
  y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n}
\end{align*}
\]

for \( n=0,1,2,... \) \quad (2)

where the parameters and initial terms are non-negative numbers such that the denominators are always positive.
Pattern of Boundedness for Equations #8 and #23: [CKLL]

The solutions of System (2) have the boundedness characterization

(U,U) if and only if
\[ \beta_2 > 0 \text{ and } B_2 = 0 \]
or
\[ \beta_2 = B_2 = C_2 = 0 \text{ and } \gamma_2 > 0 \]

(U,B) if and only if
\[ \beta_2 = B_2 = \gamma_2 = A_2 = 0 \]
or
\[ \beta_2 > 0, B_2 > 0, \text{ and } A_2 + C_2 > 0 \]

(B,B) otherwise.
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or
\[ \beta_2 > 0, B_2 > 0, \text{ and } A_2 + C_2 > 0 \]

(B, B) otherwise.

* Equation #8 and #23 are the only equations on the list of 49 equations to have this pattern of boundedness.
Generalizing Boundedness Patterns:

Let the $x_{n+1}$ equation be Equation #8 or #23.

$$
x_{n+1} = \frac{\alpha_1 + y_n}{x_n}, \quad \text{for } n=0,1,2,...
$$

$$
y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n}, \quad \text{for } n=0,1,2,...
$$

Extend the pattern of boundedness for Equations #8 and #23 to the system of equations below.

$$
x_{n+1} = \frac{\alpha_1 + y_n}{x_n}, \quad \text{for } n=0,1,2,...
$$

$$
y_{n+1} = g(x_n, y_n)
$$
Global Behavior of Solutions:
Global Behavior of Solutions:

System #(16,17):

\[
\begin{align*}
x_{n+1} &= \frac{\gamma_1 y_n}{A_1 + B_1 x_n} \\
y_{n+1} &= \frac{\beta_2 x_n}{A_2 + B_2 x_n}
\end{align*}
\]

, for \(n=0,1,2,...\)
Global Behavior of Solutions:

System #(16,17):

\[
\begin{align*}
x_{n+1} &= \frac{\gamma_1 y_n}{A_1 + B_1 x_n}, \\
y_{n+1} &= \frac{\beta_2 x_n}{A_2 + B_2 x_n}
\end{align*}
\]
, for \(n=0,1,2,...\)

We can normalize this and transform it into the system below:

System #(16,17):

\[
\begin{align*}
x_{n+1} &= \frac{y_n}{A_1 + x_n}, \\
y_{n+1} &= \frac{x_n}{A_2 + x_n}
\end{align*}
\]
, for \(n=0,1,2,...\)
Solutions converge to the (0,0) equilibrium.

Solutions converge to the unique positive equilibrium.

Solutions converge to a period-2 solution.
There is no positive equilibrium point. There are no periodic solutions. Every solution converges to the (0,0) equilibrium.

\[(16,17): \begin{cases} x_{n+1} = \frac{y_n}{A_1 + x_n} \\ y_{n+1} = \frac{x_n}{A_2 + x_n} \end{cases}\]
The (0,0) equilibrium point is a repeller. There are two prime period-2 points and they are saddle points with the positive $x$ and $y$ axes as their global stable manifolds. Every solution with positive initial terms will converge to the unique positive equilibrium point.

\[
\begin{align*}
A_2 &= A_1 \\
\#(16,17): & \quad \begin{cases} 
    x_{n+1} = \frac{y_n}{A_1 + x_n} \\
    y_{n+1} = \frac{x_n}{A_2 + x_n}
\end{cases}
\]

$$A_2 = \frac{1}{A_1}$$
The (0,0) equilibrium point is a repeller. The unique positive equilibrium point is a saddle point. There is a unique prime period-2 solution. Almost every solution of the system of equations converges to the unique prime period-2 solution.
Every solution that does not start at \((0,0)\) will converge to a period-2 solution whose components lie on the purple line segment. If \(x_0 \neq y_0\), then the solution will converge to a prime period-2 solution.

\[
\begin{align*}
\begin{cases}
x_{n+1} &= \frac{y_n}{A_1+x_n} \\
y_{n+1} &= \frac{x_n}{A_1+x_n}
\end{cases}
\end{align*}
\]
[CKLL] 
Let the $x_{n+1}$ equation be either equation #8 or #23 on the list.

Let $\beta_2 > 0$ and $B_2 > 0$.

Let $\{x_n, y_n\}$ be a solution of the system of equations

\[
\begin{align*}
x_{n+1} &= \frac{\alpha_1 + y_n}{x_n} \\
y_{n+1} &= \frac{\alpha_2 + x_n + \gamma_2 y_n}{A_2 + x_n + C_2 y_n}
\end{align*}
\]

(3)

where the parameters and initial terms are non-negative numbers such that the denominators are always positive.
Period-2 Trichotomy for Equation #8 and #23:

Conjecture: [CKLL]

\[ \{x_n, y_n\} \text{ converges if and only if } A_2 + C_2 < \alpha_2 + \gamma_2 \]

\[ \{x_n, y_n\} \text{ converges to a period-2 solution if and only if } A_2 + C_2 = \alpha_2 + \gamma_2 \]

There exist unbounded solutions if and only if

\[ A_2 + C_2 > \alpha_2 + \gamma_2 \]
Open Problem: [CKLL]

Find the necessary and sufficient conditions for the system of equations in Problem (1) to have a period-2 trichotomy.

\[
\begin{align*}
    x_{n+1} &= \frac{\alpha_1 + \beta_1 x_n + \gamma_1 y_n}{A_1 + B_1 x_n + C_1 y_n}, \text{ for } n=0,1,2,... \\
    y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n}
\end{align*}
\]

The parameters and initial terms are non-negative numbers such that the denominators are never equal to zero.
References:


[M]

[CL]

[KM]

[LP]

[CDL]

[CKLL]
[BCLL]
A. M. Brett, E. Camouzis, G. Ladas, C. D. Lynd,

[CLW]

[KM]