MTH 436
Real Analysis II, Spring 2010

EXAM I — Take home
Due March 8, 2010 at 3:00 PM

Name:

Instructions: Solve 5 problems (4 from Differentiation: problems 1 – 6 and ONE from Integration: problems 7 – 8). Remember that you need to work solo (alone) in this exam. Copying problems from some website in the Internet is not allowed.

(1) Let $f : I \to \mathbb{R}$ be differentiable at $c \in I$. Establish the Straddle Lemma: Given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if u, $v \in I$ satisfy

$$c - \delta(\epsilon) < u \le c \le v < c + \delta(\epsilon),$$

then we have $|f(v) - f(u) - (v-u)f'(c)| \le \epsilon(v-u)$.
[**Note:** For hint look at page 167 of the book problem 17.

(2) Suppose f is real, three times differentiable function on $[-1, 1]$, such that

$$f(-1) = 0, \qquad f(0) = 0, \qquad f(1) = 1 \qquad f'(0) = 0$$

Prove that $f^{(3)}(x) \ge 3$ for some $x \in (-1, 1)$.
Note that equality holds for $\frac{1}{2}(x^3 + x^2)$
Note. Use Taylor's Theorem, with $\alpha = 0$ and $\beta = \pm 1$, to show that there exists $s \in (0, 1)$ and $t \in (-1, 0)$ such that

$$f^{(3)}(s) + f^{(3)}(t) = 6$$

(3) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b). Show that if $\lim_{x \to a} f'(x) = A$, then $f'(a)$ exists and equals A.
Hint: Use the definition of $f'(a)$ and the Mean Value Theorem.

(4) Suppose a and c are real numbers, $c > 0$, and f is defined on $[-1, 1]$ by

$$f(x) = \begin{cases} x^a \sin(x^{-c}), & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

Prove the following statements:
a). f is continuous if and only if $a > 0$
b). $f'(0)$ exists if and only if $a > 1$
c). f' is bounded if and only if $a \ge 1 + c$
d). f' is continuous if and only if $a > 1 + c$
e). $f''(0)$ exists if and only if $a > 2 + 2c$
f). f'' is bounded if and only if $a \ge 2 + 2c$

g). f'' is continuous if and only if $a > 2 + 2c$

(5) Suppose f is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \le A|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.
Hint: Fix $x_0 \in [a, b]$, let

$$M_0 = \sup |f(x)|, \qquad M_1 = \sup |f'(x)|$$

for $a \le x \le x_0$. For any such x,

$$|f(x)| \le M_1(x_0 - a) \le A(x_0 - a)M_0.$$

Hence $M_0 = 0$ if $A(x_0 - a) < 1$. That is, $f = 0$ on $[a, x_0]$. Proceed.

(6) Let the function $f : [0, 1] \to [0, 1]$ be continuous on $[0, 1]$ and differentiable on $(0, 1)$. We know that there is *at least one* point in $c \in [0, 1]$ for which $f(c) = c$. Use Rolle's Theorem to prove, that if $f'(x) \neq 1$ in $(0, 1)$, then there is *exactly one* such point c.

You need to choose ONE of the following problems

(7) (a) Write down functions f and g on $[0, 1]$ such that f is integrable, g is not integrable, and fg is integrable.
(b) Write down a function f on $[0, 1]$ such that $|f|$ is integrable but f is not integrable on $[0, 1]$.

(8) Prove that, if the functions f and g are integrable on $[a, b]$, then so is the function $\max\{f, g\}$