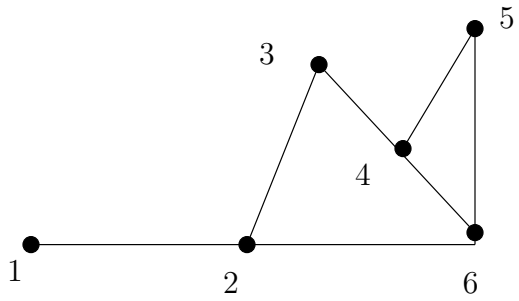
Project I
Adjacency Matrix of a Graph
Proposed by Douglas B. Meade
Due October 29, 2004

You can work in teams of 2 or 3 people.

Definition 0.1. A graph *is a finite set of objects called nodes, together with some paths between some of the nodes, as illustrated below.* **A path of length one** *is a path that directly connects one node to another.***A path of length** k *is a path made up of* k *consecutive paths of length one. The same length one path can appear more than once in a longer path; for example: 1-2-1 is a path of length two from node 1 to itself in the example below.*

When the nodes are numbered from 1 to n, the *adjacency matrix* A of the graph is defined by letting $a_{ij} = 1$ if there is a path of length one, i.e., a direct path, between vertices i and j and $a_{ij} = 0$ otherwise.

Example: Verify that matrix A is the adjacency matrix for the graph shown below.



$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

THEOREM 0.2. *(***Interpretation of the powers of an adjacency matrix***) If* A *is the adjacency matrix of a graph, then the* (i, j) *entry of* A^k *is a non-negative integer which is the number of paths of length* k *from node* i *to node* j*.*

(1) To understand why the theorem is true, compute **BY HAND** the a_{63}^2 entry of A^2. Using multiplication of matrices. Use the following table to complete this computation:

Term	$a_{61}a_{13}$	$a_{62}a_{23}$	$a_{63}a_{33}$	$a_{64}a_{43}$	$a_{65}a_{53}$	$a_{66}a_{63}$	a_{63}^2 entry of A^2
Explicit Product		(1)(1)					
Simplified Product		1					

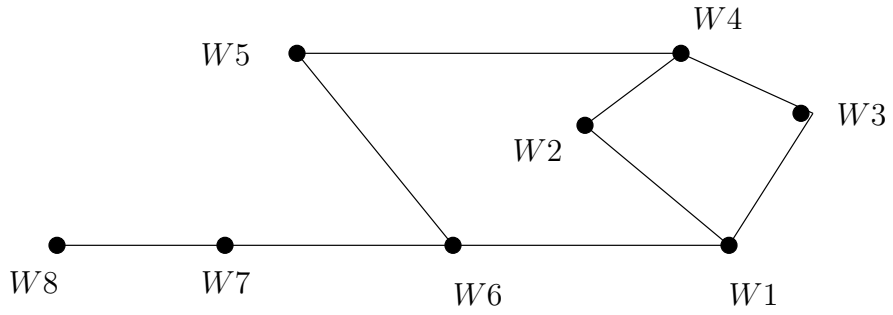
(2) Observe that the product $a_{62}a_{23} = (1)(1) = 1$ says that there is one length two path connecting nodes 6 and 3 (the intermediate node is node 2). Explain what each of the remaining five terms in the sum for the a_{63}^2 entry of A^2 tells about paths of length 2 from node 6 to node 3.

(3) Use maple to:
   (a) Find A^2 and A^3.
   (b) What can you say about the entry a_{12}^2 of A^2 ?. How many paths of length two are there from node 1 to node 2? Verify your answer studying the graph. What can you say about the entry a_{66} of A^3? How many paths are there of length three from node 6 to itself? Describe these paths.
      (i) How many paths of length two go from node 4 to itself? What are they?
      (ii) How many paths of length three go from node 4 to node 5? What are they?
      (iii) Which pair(s) of nodes are connected with the most paths of length 2?. How many?
      (iv) Which pair(s) of nodes are not connected by any path of length 2 or 3? What are they?

   Definition 0.3. *A graph is said to have* **contact level** k *between node* i *and node* j *if there is a path of length less than or equal to* k *from node* i *to node* j*.*

   (c) Suppose A is the adjacency matrix of a graph. Explain why you must calculate the sum $A + A^2 + \ldots + A^k$ in order to decide which pairs of nodes have contact level k?

(4) Eight workers, denoted $W1, \ldots W8$, handle a potentially dangerous substance. Safety precautions are taken, but accidents do happen occasionally. It is known that if a worker becomes contaminated, s/he could spread this through contact with another worker. The graph below shows which workers have direct contact with which others.
   (a) Write the adjacency matrix A for the following graph:



   (b) Enter the adjacency matrix A in your Maple session and answer the following questions:
      (i) Which workers have contact level 3 with $W3$?
      (ii) Which workers have contact level 3 with $W7$?
   (c) What is the smallest k such that every worker has contact level k with every other worker? Explain how you know your answer is correct. (Hint: Use Maple to examine A, $A + A^2$, $A + A^2 + A^3$, etc.)

(d) Define what is meant when a worker is **dangerous**. Be very specific so anyone could decide whether a worker was **dangerous** according to your definition.

(e) Using your definition, answer the following questions. Be sure to explain your answers and verify that they are consistent with your definition of **dangerous**.

(i) Which workers are the most dangerous if contaminated?

(ii) Which workers are less dangerous if contaminated?