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ATTRACTORS

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Abstract. In this paper we investigate attractors that are extended in space, but where the internal dynamics is ignored.

This paper is dedicated to Professor Hans Grauert

1. Introduction

Attractors play an important role in dynamics. The most basic ones are attracting fixed points (or periodic points). There has also been a lot of investigation about attractors that are extended in space. In that case, the focus has been on the investigation of detailed dynamical properties inside the attractor. See Lorenz ([L]), Tucker ([T]), Hénon ([H]), Bendedicks-Carleson ([BC]), Fornæss-Gavosto ([FG]), Fornæss-Weickert ([FW]), Jonsson-Weickert ([JW]) and Fornæss-Sibony ([FS2]). In this paper we investigate attractors that are extended in space, but where the internal dynamics is ignored.

Let M be a topological space and $f : M \to M$ a continuous map. Let ρ be a continuous pseudometric on M and let μ be a probability measure on M. An *attractor*, in this paper, is a distinguished point p in M which is said to *absorb* all points $q \in M$ whose orbit get closer than a given number, called *radius*, $\epsilon_p > 0$, i.e., $\rho(f^n(p), f^n(q)) \leq \epsilon_p$ for some integer $n \geq 0$. The basin of attraction, $B(p)$, of an attractor p consists of those q which gets absorbed by p. We will say that such q collide with p. We can also call such an attractor a collision-attractor, a sticky attractor or a planetary attractor. One might suggestively think of asteroids colliding with planets. Gravitational attraction makes the asteroid stick to the planet. More generally, structures grow by adding ingredients.

In Section 2 we consider the case of an invariant mixing ergodic measure and prove that all attractors absorb mass at the same rate. In Section 3, we show for hyperbolic maps with product structure that a dense set of points avoid any given attractor. In Section 4 we extend the result of Section 3 to arbitrary holomorphic maps of $\mathbb{P}^k$. We would like to thank the referee for many helpful comments.

2. Endomorphisms

Let (M, σ) be a compact metric space M, with metric σ and $f : M \to M$ a continuous map. Let ρ be a continuous pseudometric on M. We assume that M carries an invariant, mixing Borel probability measure μ. So $\mu(E) = \mu(f^{-1}(E))$ and

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$$c_n(E,F) := \mu(f^{-n}(E)\cap F) \to_{n\to\infty} \mu(E)\mu(F)$$

for all measurable sets E, F. For $x \in M, \epsilon > 0, n \geq 0$ let

$$\begin{aligned}
C_\rho(x,\epsilon,n) &:= \{y \in M; \rho(f^n(x), f^n(y)) \leq \epsilon\}\\
\mathcal{C}_\rho(x,\epsilon,n) &:= \cup_{k\leq n} C_\rho(x,\epsilon,k)\\
\mathcal{C}_\rho(x,\epsilon) &:= \cup_n \mathcal{C}_\rho(x,\epsilon,n)\\
\mathcal{C}_\rho(x) &:= \cap_{\epsilon>0}\mathcal{C}_\rho(x,\epsilon).
\end{aligned}$$

So $\mathcal{C}_\rho(x,\epsilon)$ is the basin $B(x)$ of x with the given radius, $\epsilon_x = \epsilon$.

PROPOSITION 2.1. *Let $0 < \zeta < 1$. There exists an integer $N = N(\epsilon,\zeta)$ so that if $x \in S_\mu$, the support of μ, i.e., the smallest compact set K for which $\mu(K^c) = 0$, then $\mu(\mathcal{C}_\rho(x,\epsilon,N)) \geq \zeta$.*

Proof: It suffices to prove the Proposition in the case when the pseudometric ρ is the given metric σ of M since those basins are smaller after changing ϵ. We cover S_μ by finitely many open balls $V_i = B(x_i, \epsilon/2)$ each of which intersects S_μ. Set $\eta = \min_i \mu(V_i) > 0$. For $j \geq 0$, let P_j denote the following statement:

P_j**:** There exist an integer $n_j \geq 1$, finitely many open sets $\{U^j_\ell\}_\ell$ and compact sets $\{F^j_\ell\}_\ell$ in M such that
(i) $S_\mu \subset \cup_\ell U^j_\ell$
(ii) $\mu(F^j_\ell) \geq 1 - \left(1 - \frac{\eta}{2}\right)^j$
(iii) If $x \in U^j_\ell$ then $\mathcal{C}_\rho(x,\epsilon,n_j) \supset F^j_\ell$.

The statement P_0 is trivially satisfied with $U^0_1 = M$, $F^0_1 = \emptyset$ and $n_0 = 1$.

Suppose that we have proved $P_j, j \geq 0$. We prove P_{j+1}. For each i, ℓ we have from the mixing property of μ that

$$\begin{aligned}
\mu(f^{-n}(V_i)\cap(M\setminus F^j_\ell)) &\to_{n\to\infty} \mu(V_i)\mu(M\setminus F^j_\ell)\\
&= \mu(V_i)(1-\mu(F^j_\ell)).
\end{aligned}$$

Hence we can find $n_{j+1} > n_j$ so that for all i, ℓ

$$\mu(f^{-n_{j+1}}(V_i)\cap(M\setminus F^j_\ell)) \geq \eta(1-\mu(F^j_\ell)) - \delta$$

where $\delta > 0$ will be fixed later. (See formula (*) below.)

Next let $K^j_{i,\ell} \subset M\setminus F^j_\ell$ be a compact subset of $f^{-n_{j+1}}(V_i)\cap(M\setminus F^j_\ell)$ so that

$$\mu(K^j_{i,\ell}) \geq \eta(1-\mu(F^j_\ell)) - 2\delta.$$

Define next $F^{j+1}_{i,\ell} = F^j_\ell \cup K^j_{i,\ell}$. Also define $U^{j+1}_{i,\ell} = f^{-n_{j+1}}(V_i)\cap U^j_\ell$. We show that $\{U^{j+1}_{i,\ell}, F^{j+1}_{i,\ell}\}_{i,\ell}$ satisfy $(i),(ii),(iii)$ except for reordering.

(i) If $x \in S_\mu$, then by $P_j, x \in U^j_\ell$ for some ℓ. Also $f^{n_{j+1}}(x) \in S_\mu \subset \cup V_i$. Hence $f^{n_{j+1}}(x) \in V_i$ for some i. Therefore, $x \in U^{j+1}_{i,\ell}$. So $S_\mu \subset \cup U^{j+1}_{i,\ell}$.

(iii) If $x \in U^{j+1}_{i,\ell}$, then $x \in U^j_\ell$ so $\mathcal{C}_\rho(x,\epsilon,n_{j+1}) \supset \mathcal{C}_\rho(x,\epsilon,n_j) \supset F^j_\ell$. Also $x \in U^{j+1}_{i,\ell} \Rightarrow f^{n_{j+1}}(x) \in V_i$. If $y \in K^j_{i,\ell}$, then $y \in f^{-n_{j+1}}(V_i)$ so $f^{n_{j+1}}(y) \in V_i$ also. Hence $\rho(f^{n_{j+1}}(x), f^{n_{j+1}}(y)) < \epsilon$ which implies that $y \in \mathcal{C}_\rho(x,\epsilon,n_{j+1})$. Hence $\mathcal{C}_\rho(x,\epsilon,n_{j+1}) \supset F^j_\ell \cup K^j_{i,\ell} = F^{j+1}_{i,\ell}$.

(ii) Since $K^j_{i,\ell} \subset M \setminus F^j_\ell$, we have

$$\begin{aligned}
\mu(F^{j+1}_{i,\ell}) &= \mu(F^j_\ell \cup K^j_{i,\ell}) \\
&= \mu(F^j_\ell) + \mu(K^j_{i,\ell}) \\
&\geq \mu(F^j_\ell) + \eta(1-\mu(F^j_\ell)) - 2\delta \\
&= (\eta - 2\delta) + \mu(F^j_\ell)(1-\eta) \\
&\geq \eta - 2\delta + (1-(1-\frac{\eta}{2})^j)(1-\eta/2-\eta/2) \\
&= 1-(1-\frac{\eta}{2})^{j+1} + \frac{\eta}{2}(1-\frac{\eta}{2})^j - 2\delta
\end{aligned}$$

So we just have to choose

$$(*)\ \delta = \frac{\eta}{4}(1-\frac{\eta}{2})^j.$$

This proves the inductive hypothesis P_{j+1}.

To complete the proof of the proposition, choose $j \geq 0$ such that $(1-\frac{\eta}{2})^j \leq 1-\zeta$ and put $N = n_j$. Then, if $x \in S_\mu$, by $P_j(i)$, $x \in U^j_\ell$ for some j, ℓ. Hence by $P_j(iii), \mu(\mathcal{C}_\rho(x,\epsilon,n_j)) \geq \mu(F^j_\ell)$ and by $P_j(ii), \mu(F^j_\ell) \geq 1-(1-\eta/2)^j \geq \zeta$. Therefore $\mu(\mathcal{C}_\rho(x,\epsilon,N)) = \mu(\mathcal{C}_\rho(x,\epsilon,n_j)) \geq \zeta$ as desired.

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COROLLARY 2.2. *For every $x \in S_\mu$, $\mu(\mathcal{C}_\rho(x)) = 1$.*

Proof: For every $\epsilon > 0$, $\mathcal{C}_\rho(x,\epsilon) \supset \mathcal{C}_\rho(x,\epsilon,N(\epsilon,\zeta))$ for all $0 < \zeta < 1$. Hence $\mu(\mathcal{C}_\rho(x,\epsilon)) = 1$. If $\epsilon_1 < \epsilon_2$, then $\mathcal{C}_\rho(x,\epsilon_1) \subset \mathcal{C}_\rho(x,\epsilon_2)$ and hence $\mathcal{C}_\rho(x) = \cap_{n=1}^\infty \mathcal{C}_\rho(x,1/n)$ so $\mu(\mathcal{C}_\rho(x)) = 1$.

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Let μ^2 denote the product measure $\mu \times \mu$ on $M \times M$.

COROLLARY 2.3. *Let $\epsilon > 0$. Under the conditions of Proposition 2.1, the set of $(p,q) \in M \times M$ which collide, i.e., $\rho(f^n(p), f^n(q)) \leq \epsilon$ for some n, has full μ^2 measure.*

Proof: For any $p \in S_\mu$, $\mu(\mathcal{C}_\rho(p,\epsilon)) = 1$. All points $q \in \mathcal{C}_\rho(p,\epsilon)$ collide with p. Integrating over p, the result follows.

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We finish this section by giving a few examples.

In the first two examples the map is ergodic but not mixing. The last three examples are all mixing, and attractors differ in whether their basins cover all of S_μ.

The irrational rotation of the unit circle, $f : T \to T$, $f(z) = ze^{i\psi}$ for a $e^{i\psi}$ not a root of unity, is ergodic for the arc-length measure $\mu = \frac{d\theta}{2\pi}$.

Example 2.4. *We choose first the pseudometric $\rho_1(z, w) = |\Re(z - w)|$. For any attractor $p \in T$ and any $\epsilon > 0$, $B(p) = T$. Physically one can think of the time-1 map of a harmonic oscillator where $\Re(z)$ represents position and $\Im(z)$ represents momentum. Any $q \in T$ eventually collides with p, i.e., gets closer than $\rho_1 = \epsilon$, so the basin of p is the whole circle.*

Example 2.5. *However, for the metric $\rho_2(z, w) = |z - w|$, the basin is never the whole circle if ϵ is small. Physically this represents time-1 maps of particles in circular motion.*

Example 2.6. *If we use the pseudometric ρ_1 above for the mixing map $f = z^2$ on the circle, we also get that for small $\epsilon > 0$, the basin of q will not contain all points.*

Example 2.7. *Let $N \geq 1, X = \{x = (x_n)_{n=0}^{\infty}, x_n \in \{0, 1\}, \prod_{k=0}^{N} x_{n+k} = 0 \ \forall n\}$. We define a metric on X, $\sigma(x, y) = \sum_{n=0}^{\infty} \frac{|x_n - y_n|}{2^n}$. The shift map $f : X \to X, f(x_0, x_1, \dots) = (x_1, x_2, \dots)$ is continuous and surjective. We choose a pseudo-metric $\rho(x, y) = |x_0 - y_0|$. Then we have that for any $0 < \epsilon < 1, \mathcal{C}_\rho((0, 0 \dots), \epsilon, N + 1) = X$.*

Example 2.8. *We can construct a compact set K in the plane and a continuous self-map f with an ergodic mixing probability measure μ with support K. Moreover, each point is an attractor which absorbs all points.*
We proceed inductively. Suppose that K_n is a finite set $\{p_j\}_{j=1}^N$ and F_n is a permutation. Let $C_j, j = 1, \dots, N$ be small circles centered at p_j. The radii will shrink very rapidly with n. Fix a number $\tilde{N} > N$ which is relatively prime with N and let $M >> \tilde{N} \times N$. We put M equidistributed points $\{p_{j,k}\}_{k=1}^M$ around each p_j except around p_1 where we use $\{p_{1,k}\}_{k=1}^{M+\tilde{N}}$. Next we define the compact set $K_{n+1} := \{p_{j,k}\}$ and define F_{n+1} by letting

$$\begin{aligned}
F_{n+1}(p_{j,k}) &= p_{j+1,k}, \ j = 1, \dots, N-1, k = 1, \dots, M \\
F_{n+1}(p_{N,k}) &= p_{1,k+1}, \ k = 1, \dots, M \\
F_{n+1}(p_{1,k}) &= p_{1,k+1}, \ k = M+1, \dots, M+\tilde{N}-1 \\
F_{n+1}(p_{1,M+\tilde{N}}) &= p_{1,1}
\end{aligned}$$

Any given two points on the limiting compact set have orbits that come arbitrarily close to each other.

In Figure 1 we give pictures of an attractor after 2 and 10 iterations.

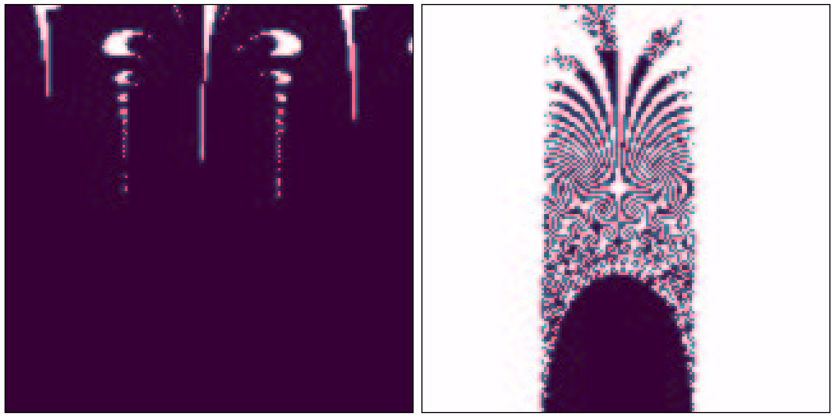

FIGURE 1. Attractor for $f(z) = \exp(i\psi z)$. The picture (left) shows the basin of 1 in white after 2 iterates. The picture (right) shows the attractor after 10 iterates. In both cases $\psi = .7$ and $\epsilon = 0.5$. The pseudometric is $|\Re z - \Re w|$. The rectangles are [-8, 4]x [-14, -2]

3. Hyperbolic Diffeomorphisms

Suppose that M is a smooth manifold with Riemannian metric $\rho = ds$. Let $f : M \to M$ be a smooth $\mathcal{C}^\infty$ diffeomorphism. Assume that K is a compact totally invariant perfect set. Suppose that f is hyperbolic on K with continuously varying stable subbundle E^s and unstable bundle E^u. We assume that $\dim E^s, \dim E^u > 0$ and that K has local product structure.

PROPOSITION 3.1. *If $p \in K$, then $K \setminus \mathcal{C}_\rho(p)$ is dense in K.*

Proof: Suppose $p \in K$. Pick $q \in K \setminus \{p\}$. We want to show that there are points in K arbitrarily close to q which doesn't belong to $\mathcal{C}_\rho(p)$. Let $0 < \tau << 1$ be arbitrary and let $W^u_\tau(q)$ denote the local unstable manifold of q with radius τ. It suffices to find an integer n so that $W^u_\tau(q) \setminus \mathcal{C}_\rho(p, 1/n) \neq \emptyset$.

We start by writing down estimates that follow from the hyperbolic local product structure. There exist constants $\delta, \eta > 0, \Lambda, \lambda > 1$ and an integer $N > 1$ so that whenever $x', y' \in W^u_\delta(z'), z' \in K$ and $f^j(x'), f^j(y') \in W^u_\delta(f^j(z')), j = 1, \ldots, m$ then

$$\begin{aligned}
\rho(f^j(x'), f^j(y')) &\geq \eta\rho(x', y') \;\forall\, j \leq N \\
\rho(f^j(x'), f^j(y')) &> \lambda^j \rho(x', y'),\ N < j \leq m \\
\rho(f^j(x'), f^j(y')) &\leq \Lambda^j \rho(x', y'), 1 \leq j \leq m
\end{aligned}$$

(*) Moreover, whenever $x'', z'' \in K$, $r > 0$ and $x'' \in W^u_{\delta/2}(z'')$ with $W^u_r(x'') \subset W^u_\delta(z'')$ there is a $y'' \in (W^u_r(x'') \setminus W^u_{\eta r}(x'')) \cap K$.(*)

We will find an arbitrarily small $\tau > 0, \tau >> \epsilon > 0$ and sequences $\{q_k\}_{k=0}^\infty \subset W^u_\tau(q), \{n_k\}_{k=0}^\infty \subset \mathbb{Z}^+, 0 \leq n_0 < n_1 < \cdots$ so that $f^{n_k}(q_\ell) \in W^u_\tau(f^{n_k}(q_k)), \ell > k$

and $\mathcal{C}_\rho(p,\epsilon,n_k)\cap f^{-n_k}(W^u_\tau(f^{n_k}(q_k)))=\emptyset$. Then $\lim_{k\to\infty} q_k$ is in $\overline{W^u_\tau(q)}$ but is not in $\mathcal{C}_\rho(p)$ and the proof will be complete.

The construction of the sequences $\{q_k\},\{n_k\}$ goes by induction. The requirements needed for ϵ,τ will become evident during the proof. First of all we need ϵ,τ small enough that $p\notin W^u_\tau(q)$ to start the induction. Let $n_0=0, q_0=q$. Suppose n_k,q_k have been chosen and that $\mathcal{C}_\rho(p,\epsilon,n_k)\cap f^{-n_k}(W^u_\tau(f^{n_k}(q_k)))=\emptyset$. There are two cases:

(I) $\mathcal{C}_\rho(p,\epsilon,n_k+1)\cap f^{-n_k-1}(W^u_\tau(f^{n_k+1}(q_k)))=\emptyset$. Then we define $n_{k+1}=n_k+1$ and set $q_{k+1}=q_k$.

(II) $\mathcal{C}_\rho(p,\epsilon,n_k+1)\cap f^{-n_k-1}(W^u_\tau(f^{n_k+1}(q_k)))\neq\emptyset$. Then there is a point $x'\in W^u_{2\tau}(f^{n_k+1}(q_k))$ so that $x:=f^{n_k+1}(p)\in W^s_{loc}(x')$ and $\rho(x,x')\le C\epsilon$ for some fixed constant C where C depends on the angle between the stable and unstable manifolds. Set $y:=f^{n_k+1}(q_k)$. We consider two cases:

(IIA) $\rho(x',y)\ge\frac{C\epsilon}{\eta}$

Then the distance between $f^j(x)$ or $f^j(x')$ and $f^j(y)$ remains strictly larger than ϵ for $j\le N$ and for $j>N$, $\rho(f^j(x),f^j(y)),\rho(f^j(x'),f^j(y))\ge\lambda^j\rho(x',y)$, as long as $\rho(x',y)\Lambda^j\le\delta$ i.e., $j\log\Lambda\le\log\frac{\delta}{\rho(x',y)}$ or
$j\le\frac{1}{\log\Lambda}\log\left(\frac{\delta}{\rho(x',y)}\right)$.

Assume first that $N\le\frac{1}{\log\Lambda}\log\left(\frac{\delta}{\rho(x',y)}\right)$, i.e.,

(IIA1): $\rho(x',y)\le\frac{\delta}{\Lambda^N}$.

For $j_0=\left[\frac{1}{\log\Lambda}\log\left(\frac{\delta}{\rho(x',y)}\right)\right]$ we have

$$\rho(f^{j_0}(x'),f^{j_0}(y))\ge\rho(x',y)\lambda^{\left[\frac{1}{\log\Lambda}\log\left(\frac{\delta}{\rho(x',y)}\right)\right]}.$$

In fact, we can assume that no point in

$$W^u\left(f^{j_0}(y),\frac{\rho(x',y)}{2}\lambda^{\left[\frac{1}{\log\Lambda}\log\left(\frac{\delta}{\rho(x',y)}\right)\right]}\right)$$

have been captured by p for $0\le j\le j_0$, i.e., points in

$$W^u(f^{j_0}(y),\frac{1}{2}\rho(x',y)^{1-\frac{\log\lambda}{\log\Lambda}}\delta^{\frac{\log\lambda}{\log\Lambda}})$$

have not been captured.

Notice that if $\epsilon>0$ is small enough, $\frac{1}{2}\left(\frac{C\epsilon}{\eta}\right)^{1-\frac{\log\lambda}{\log\Lambda}}\delta^{\frac{\log\lambda}{\log\Lambda}}>\epsilon^\sigma,\sigma:=1-\frac{1}{2}\frac{\log\lambda}{\log\Lambda}$ and moreover, $\epsilon^\sigma<<\frac{\delta}{N^\Lambda}$. We set $\tau=\epsilon^\sigma, n_{k+1}=n_k+j_0$ and pick some $q_{k+1}\in W^u_{\tau/2}(f^{n_k+j_0}(q_k))\cap K$.

(IIA2): $\rho(x',y)>\frac{\delta}{\Lambda^N}$. This can't happen since $\tau<<\frac{\delta}{\Lambda^N}$ and $x'\in W^u_{2\tau}(y)$.

Next we deal with

(IIB): $\rho(x',y)<\frac{C\epsilon}{\eta}$.

Using (*) we can find a $y' \in W^u_\delta(y)$ with $\frac{2C\epsilon}{\eta} \leq \rho(y', y) \leq \frac{2C\epsilon}{\eta^2}$. Replacing y by y' we are back to case (IIA).

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4. Holomorphic Endomorphisms of $\mathbb{P}^k$

Holomorphic endomorphisms on $\mathbb{P}^k$ carry a unique invariant measure μ, ergodic, mixing and of maximal entropy ([FS1],[BD2]).

THEOREM 4.1. *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic map of degree d at least 2. Let ρ denote the Fubini-Study metric. Then for any $q \in S_\mu$ the complement of $\mathcal{C}_\rho(q)$ in S_μ is dense in S_μ.*

Proof: Fix $q \in S_\mu$. Let $p \in S_\mu$ and let $\epsilon_1 > 0$. We need to find an $\epsilon > 0$ and a point $p'' \in S_\mu \cap \Delta(p, \epsilon_1) = \{x \in S_\mu; \rho(x, p) < \epsilon_1\}$ with $p'' \in \mathcal{C}_\rho(q, \epsilon)^c$.
(i) Suppose first that $\mathcal{O}(q)$ is a finite set S. By ([BD1]) the repelling periodic orbits are dense in S_μ, hence we can find a periodic point $p' \in S_\mu \cap \Delta(p, \epsilon_1)$ not in S. Hence $\mathcal{O}(p') \cap \mathcal{O}(q) = \emptyset$ and since they are finite, we can choose $0 < \epsilon < \rho(\mathcal{O}(q), \mathcal{O}(p'))$ and obtain that $p' \in \mathcal{C}_\rho(q, \epsilon)^c$.
(ii) Suppose that $\mathcal{C}(q)$ is infinite. We argue using homoclinic orbits. Pick an integer $\ell >> 1$. Let C denote the critical set of f. Set $C_\ell := C \cup f(C) \cup \cdots \cup f^\ell(C)$. For a generic point $x \in \mathbb{P}^k$ the measures $\mu^i_x := \frac{1}{d^{ki}} \sum_{y \in f^{-i}(x)} \delta_y$ converge weakly to μ. Here the points y are counted with multiplicity. In particular this holds for all x outside an algebraic subvariety of zero $\mu-$ mass. This subvariety is contained in C ([BD2]). Since algebraic varieties carry no $\mu-$mass ([FS1], [BD2]) and since repelling periodic orbits are dense in S_μ ([BD1]), we can find a point $p_0 \in [\Delta(p, \epsilon_1) \cap S_\mu] \setminus C_\ell$ which is on a repelling periodic orbit $\{f^j(p_0) =: p_j\}_{j=0}^N, p_N = p_0$ contained in S_μ.

For each $i \geq 1$, let $T^i = \{p^i_j\}_{j=1}^{d^{ki}}$ denote all the f^{-1} preimages of p_0 counted with multiplicity. Since S_μ is totally invariant ([FS1]), all preimages are contained in S_μ. The measures $\mu^i := \frac{1}{d^{ki}} \sum_{j=1}^{d^{ki}} \delta_{p^i_j}$ converge weakly to μ.

Let p^1_1 denote a preimage of p_0 which is not on the periodic orbit of p_0 and let p^2_1, p^2_2 denote two distinct preimages of p^1_1. For $i \geq 2$, let $T^i_s \subset T^i, s = 1, 2$ denote the set of points p^i_j for which none of the points $p^i_j, f(p^i_j), \ldots, f^i(p^i_j)$ are on the critical set and $f^{i-2}(p^i_j) = p^2_s$.

Hence when $i \leq \ell$, T^i_s contains exactly $d^{k(i-2)}$ points, each of which has multiplicity one. Therefore, if $i \leq \ell, s = 1, 2$

$$\mu^i_s := \frac{1}{d^{ki}} \sum_{x \in T^i_s} \delta_x$$

has mass $\frac{1}{d^{2k}}$. However, for $i > \ell$ some preimages $f^{-i}(p)$ might be on the critical set. Hence the masses decrease, $\|\mu^{i+1}_s\| \leq \|\mu^i_s\|$.

By a counting argument as in ([BD1]), based on Bezouts theorem, we can, assuming that ℓ is large enough, have $\|\mu^i_s\| \geq \frac{1}{2d^{2k}}$ for all i, s.

Let $U = U(p_0)$ be a small neighborhood of p_0, $U \subset \Delta(p, \epsilon_1)$, on which f^N is biholomorphic onto its image and $f^{-N}(U) \cap U \subset\subset U$. We can assume f^N is strictly expanding in some local coordinate system, i.e., $\|f^N(x) - f^N(y)\| \geq \lambda\|x-y\| \; \forall \, x, y \in U, \lambda > 1$ a constant. Since p_0 is in the support of μ, $\mu(U) =: \delta > 0$. From ([FS1]) it follows that for large enough r, $\mu(f^r(U)) \geq 1 - \frac{1}{4d^{2k}}$

Set $C^r := C \cup_{i=1}^r f^{-i}(C)$. Then C^r has no $\mu-$ mass and p_0 is not in C^r.

Set $V_0 = U \setminus C^r$ and $V_i = f^i(V_0), i = 1, \ldots, r$. Then $\{f^i\}$ are locally biholomorphic on V_0 and $\mu(V_r) = \mu(f^r(U)) \geq 1 - \frac{1}{4d^{2k}}$.

By the estimate of $\|\mu_s^i\|$, it follows that for large enough i_0, $\mu_s^{i_0}(V_r) > 0, s = 1, 2$. Therefore there exists two points $\tilde{p}_1, \tilde{p}_2 \in V_0 \cap S_\mu$ so that the orbits $\{f^i(\tilde{p}_s)\}_{i=0}^{r+i_0} \subset S_\mu$ contain no critical points and the orbits are disjoint except that $f^{r+i_0-1}(\tilde{p}_1) = f^{r+i_0-1}(\tilde{p}_2) = p_1^1$. Also $f^{r+i_0-2}(\tilde{p}_s) = p_s^2$.

Fix two small neighborhoods $W_s = W_s(\tilde{p}_s)$ so that all $f^i, i = 1, \ldots, r + i_0$ are biholomorphic on W_s, the images are all pairwise disjoint, except that $f^{r+i_0-1}(W_1) = f^{r+i_0-1}(W_2)$ and $f^{r+i_0}(W_s) = \Delta(p_0, \eta)$ for a small $\eta > 0, s = 1, 2$. Define $W_s^{-1} = f^{-1}(W_s) \cap U$ and inductively, $W_s^{-(j+1)} = f^{-1}(W_s^{-j}) \cap U$. Fix a j_0 large enough that $W_s^{-j_0} \subset \Delta(p_0, \eta/2), s = 1, 2$. Increasing j_0 further, we can also arrange that the biholomorphic maps $f^{j_0+r+i_0} : W_s^{-j_0} \to \Delta(p_0, \eta)$ is strictly expanding.

We will define a sequence of open sets $\{U_i\}_{i \geq 0}, U_{i+1} \subset U_i$, and occasionally so that $U_{i+1} \subset\subset U_i$, $f^i : U_i \to W(i)$ is a biholomorphic map where $W(i)$ is one element of the following finite list of open sets:

$\Delta(p_0, \eta)$
$f^i(\Delta(p_0, \eta)), i = 1, \ldots, N - 1$
$W_s^{-i}, i = 1, \ldots, j_0, s = 1, 2$
$W_s, s = 1, 2$
$f^i(W_s), i = 1, \ldots, r + i_0 - 1, s = 1, 2.$

Since each of these open sets contain points in S_μ, each U_i will also. We moreover want $W(i)$ not to contain a point in $\Delta(f^i(q), \epsilon)$, if $\epsilon > 0$ is small enough.

To start we will define $U_0 = \Delta(p_0, \eta)$. Since q is not in U, we can take $W(0) = U_0$. Next we continue by setting $U_1, \ldots, U_{N-1} = U_0$ and $W(i) = f^i(U_i), i = 1, \ldots, N-1$ unless $f^i(q)$ gets closer to this $W(i)$ than ϵ.

Assuming at first that $f^i(q)$ does not get closer than ϵ, we continue by setting $U_N = f^{-N}(U_0) \cap U_0$ and $W(N) = U_0$ and continue this process the same way. This procedure is only interrupted if $f^i(q)$ gets closer to these $W(i)$ than ϵ for some i. If this occurs for $i = i_1$, then we redefine U_i, call the new U_i, $\tilde{U}_i$ with $\tilde{U}_i \subset\subset U_i$ and if $i = jN + r, 0 \leq r \leq N$, then $f^i(\tilde{U}_i)$ is one of the sets $f^r(W_s^{-j_0}), s = 1, 2$. Since these two sets as well as their forward orbits until they reach p_s^2 are separated, choosing the right one of the two ensures that the orbit $f^j(q)$ stays at least an ϵ distance away until we return to p_s^2. But once we return there, we can repeat the process. Notice that by the strict expansion of the iterates f^N on U_0 and $f^{j_0+r+i_0}$ on $W_s^{-j_0}$, the diameters of the U_i shrink geometrically. Finally we set $p'' = \cap_{i>0}\overline{U}_i = \cap_{i>0}U_i$. Clearly $\rho(f^n(p''), f^n(q)) > \epsilon \; \forall \, n \geq 0$ so $p'' \in \mathcal{C}_\rho(q, \epsilon)^c$ as desired.

■

Since the measure μ is mixing, ([FS1]), we can combine the above result with Corollary 2.2 and obtain:

COROLLARY 4.2. *Let ρ denote the Fubini-Study metric on $\mathbb{P}^k$. Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic map of degree at least 2. Then for every $z \in S_\mu$, $\mathcal{C}_\rho(z)$ has full μ measure while the complement of $\mathcal{C}_\rho(z)$ is dense in S_μ.*

The results here naturally suggest the following questions:

Question 4.3. *Is it possible that all repelling periodic points of a holomorphic self-map of $\mathbb{P}^k, k \geq 2$, are in the critical orbit? In fact, given $C_\ell = C \cup f(C) \cup \cdots \cup C_\ell$, does there exist a repelling periodic orbit which does not intersect C_ℓ? (even for $\ell = 1$.)*

Question 4.4. *Is Proposition 3.1 valid for general hyperbolic endomorphisms in the case the pseudometric is a metric?*

Question 4.5. *When does Theorem 4.1 hold for rational self maps of $\mathbb{P}^2$?*

We end with a partial result on Question 4.5, namely in the case of a complex Hénon maps.

If $f : \mathbb{C}^2 \to \mathbb{C}^2$ is a Hénon map of degree $d \geq 2$, then there is a unique invariant measure μ, ergodic, mixing and of maximal entropy ([BLS]). The support of μ, S_μ, is a compact subset of $\mathbb{C}^2$. Let ρ be any continuous metric.

PROPOSITION 4.6. *Let f be a Hénon map. For any $q \in S_\mu$, the complement $\mathcal{C}_\rho(q)^c \cap S_\mu$ of $\mathcal{C}_\rho(q)$ in S_μ is dense in S_μ.*

We state first a Lemma about behaviour near saddle points. Let p be a saddle periodic point for f. We choose a local holomorphic coordinate system (z, w) near p so that

$$\begin{aligned} W^u_{p,loc} &= \{(z,w); w = 0, |z| < 1\} \\ W^s_{p,loc} &= \{(z,w); z = 0, |w| < 1\} \end{aligned}$$

are local unstable and stable manifolds respectively and $p = 0$. By ([BLS]) there is a transverse intersection of the global unstable and stable manifolds other than p.

LEMMA 4.7. *Suppose $(z_0, 0), 0 < |z_0| < 1$ is a transverse homoclinic point, i.e., $(z_0, 0) \in W^s(p)$ and the tangent space of $W^s(p)$ at $(z_0, 0)$ is not the $z-$ axis. Let $|z_0| < r < 1$, and let $0 < \delta < r - |z_0|$. Then there exists an arbitrarily small $\eta > 0$ and an integer $N > 1$ so that if $n \geq N$ and X is a holomorphic graph $w = g(z), z \in \Delta(z_0, \delta), |g| < \eta$ then $f^n(X)$ contains a relatively open set Y which is a graph $w = h(z), |z| < r, |w| < \eta$.*

Proof: Observe that the conclusion follows easily for the special case when $X = \{(z, 0); z \in \Delta(z_0, \delta)\}$. Finally we choose η small enough to complete the proof.

■

Proof of the Proposition: Suppose $p, q \in S_\mu$ and $\epsilon_1 > 0$. We will find an $\epsilon > 0$ and a point $p'' \in S_\mu \cap \Delta(p, \epsilon_1)$ such that $p'' \in \mathcal{C}_\rho(q, \epsilon)^c$.

If q is a periodic point, since periodic points are dense in S_μ ([BLS]) and points can carry no mass, we can let p'' be on another periodic orbit and we are done.

So we can suppose that the orbit of q is infinite. We choose a periodic saddle point $p' \in \Delta(p, \epsilon_1)$ which then necessarily is not on the orbit of q. Use local coordinates (z, w) as in the above Lemma near p'. We can assume that the bidisc $\{|z|, |w| < 1\} \subset \Delta(p, \epsilon_1)$. By ([BLS]) p' admits transverse homoclinic points. Hence we can find two distinct points $z_0^k \neq 0$ and disjoint discs $\overline{\Delta(z_0^0, \delta)}, \overline{\Delta(z_0^1, \delta)}$, with $z_0^k \in W^s(p'), k = 0, 1$. Let (η^k, N^k) denote the numbers from the Lemma for the two points. We can choose $r > |z_0^k| + \delta, k = 0, 1$ in both cases.

Letting $\eta = \min_k \{\eta^k\}$ and choosing $N \geq \max_k N^k$ large enough we can conclude:

LEMMA 4.8. *Suppose X^k is a graph $w = g^k(z), z \in \Delta(z_0^k, \delta), |g^k| < \eta$, then $f^N(X^k)$ contains graphs $w = h^{k,\ell}(z), z \in \Delta(z_0^\ell, \delta), |h^{k,\ell}| < \eta, k, \ell = 0, 1$.*

We fix $\epsilon_2 = \min_{i \leq N} \rho(f^i(\overline{\Delta(z_0^0, \delta)} \times (|w| < \eta)), f^i(\overline{\Delta(z_0^1, \delta)} \times (|w| < \eta)))$. Choose $\epsilon < \epsilon_2$ so that if $x \in S_\mu$ is any point and $\rho(f^i(x), f^i(\overline{\Delta(z_0^k, \delta)} \times (|w| \leq \eta))) \leq \epsilon$ for some $0 \leq i \leq N$, then $\rho(f^i(x), f^i(\overline{\Delta(z_0^{1-k}\delta)} \times (|w| \leq \eta))) > \epsilon$ for all $0 \leq i \leq N$.

We will find a point $p'' \in \cap_{i=0}^\infty U_i$ where $U_{i+1} \subset U_i \ \forall\, i, U_{i+1} \subset\subset U_i$ occasionally. Furthermore each U_i will contain a point in S_μ.

We define $U_0 = \Delta(z_0^k) \times \{0\}$ where $k \in \{0, 1\}$ is such that $\rho(f^i(q), f^i(U_0)) > \epsilon \ \forall\, i = 0, \ldots, N$. Set $U_i = U_0, i = 0, \ldots, N$. Since $(z_0^k, 0) \in W^s(p') \cap W^u(p')$ is a transverse intersection, we know ([BLS]) that $(z_0^k, 0) \in S_\mu$. Hence $U_i \cap S_\mu \neq \emptyset, i = 0, \ldots, N$.

Next, by Lemma 4.8, there are open subsets $U_N^k \subset\subset U_N, k = 0, 1$ such that $f^N(U_N^k)$ are graphs in $\Delta(z_0^k, \delta) \times (|w| < \eta)$. In particular, $f^N(U_N^k) \subset f^N(W^u(p')) = W^u(p')$, it follows that $f^N(U_N^k)$ contains a transverse homoclinic intersection so $f^N(U_N^k) \cap S_\mu \neq \emptyset$. Since S_μ is totally invariant, U_N^k contains a point in S_μ.

We repeat the process starting with $X = X^k = f^N(U_N^k)$ for $k = 0, 1$, chosen so that $\rho(f^{N+i}(q), f^i(X)) > \epsilon \ \forall\, i = 0, \ldots, N$. Also set, for this k, $U_{N+1} = \cdots = U_{2N} = U_N^k$. Proceeding inductively we obtain a sequence $\{U_i\}, U_{i+1} \subset U_i, U_{i+N} \subset\subset U_i$ and $U_i \cap S_\mu \neq \emptyset$, and $\rho(f^i(U_i), f^i(q)) > \epsilon$. We can finally choose $p'' \in (\cap \overline{U}_i) \cap S_\mu$. Then $\rho(f^i(q), f^i(p'')) > \epsilon$ for all i, so $p'' \in \mathcal{C}(q, \epsilon)^c \cap S_\mu$, as desired.

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