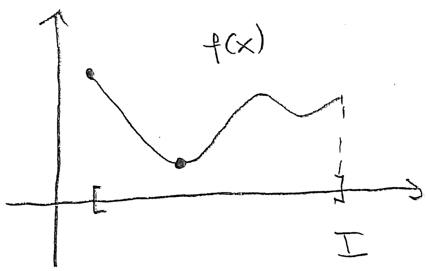
March 10

15.3 Constrained Optimization: Lagrange Multipliers

We talked about local extrema of a function $z = f(x,y)$ of two variables. In Calculus I after local extrema the subject of global extrema was discussed: find the global minimum and maximum of a given function $y = f(x)$ over a given interval I:



Global mimum and maximum were either inside I, in which case they were at critical points, or at an endpoint of I.

The problem of finding global extrema of a given function $y = f(x)$ in a given interval I made a lot of sense because this is how optimization problems for functions of one variable look in applications.

A typical optimization problem where the objective function $f(x,y)$ is a function of two variables looks as follows:

$$\begin{cases} \text{Maximize} : f(x,y) & \text{the objective function} \\ \text{Subject to} : g(x,y) \leq c & \text{constraints} \end{cases}$$

($f(x,y)$, $g(x,y)$, c - a constant given).

Maximize or minimize depending on a problem.

Such problem is called a constrained optimization problem. The main tool for solving constrained optimization problems involves Lagrange multipliers.

Another type of a constraint optimization problem:

$$\begin{cases} \text{Maximize: } f(x,y) \\ \text{Subject to : } g(x,y) = c \end{cases}$$ equality constraints

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Ex: Suppose we want to maximize production, $f(x,y)$, (in units of whatever it is that we produce), where x, y are quantities of raw materials needed. Suppose:

$$f(x,y) = x^{\frac{2}{3}} \cdot y^{\frac{1}{3}}$$

If x, y are purchased at prices p_1, p_2 thousands dollars per unit of x and y, what is the maximum production we can achive with a given budgetary limit $\underline{\underline{c}}$ thousand dollars? Here we have a typical constrained optimization problem:

$$\begin{cases} \text{Maximize: } & x^{\frac{2}{3}} \cdot y^{\frac{1}{3}} \\ \text{Subject to : } & p_1 x + p_2 y \leq c, \end{cases}$$

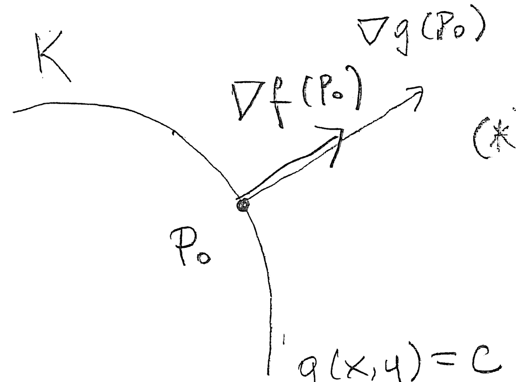
p_1, p_2, c given. Suppose that in our case $p_1 = p_2 = 1$, $c = 3.78$. So we have:

$$\begin{cases} \text{Maximize : } & f(x,y) = x^{\frac{2}{3}} \cdot y^{\frac{1}{3}} \\ \text{Subject to : } & x + y \leq 3.78 \end{cases}$$

Obviously the maximum production will be achived when we do use all the available budget. So the proble really is :

$$\begin{cases} \text{Maximize: } f(x,y) = x^{\frac{2}{3}} \cdot y^{\frac{1}{3}} \\ \text{Subject to : } x+y = 3.78 \end{cases}$$

How do we solve such problems?



$$(*) \begin{cases} \text{Maximize : } f(x,y) \\ \text{Subject to : } g(x,y) = c \end{cases}$$

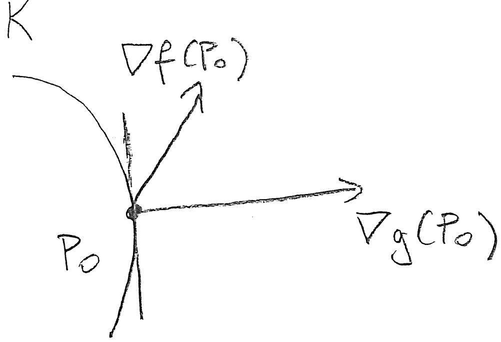
Suppose P_0 is a solution to $(*)$. What can we tell about $\nabla g(P_0)$ and $\nabla f(P_0)$? $g(x,y) = c$ is a contour curve of the function $g(x,y)$. So, assuming that every thing is smooth and gradients are not $\vec{0}$, we have :

$$\nabla g(P_0) \perp K.$$

I claim that $\nabla f(P_0)$ is also perpedicular to K:

$$\nabla f(P_0) \perp K.$$

Why? Suppose it isn't. Then $\nabla f(P_0)$ is not perpendicular to the tangent line to K at P_0:



That means that we can move from P_0 along K and increase $f(x,y)$ as we are moving a bit in the direction of $\nabla f(P_0)$. (Remember $f_{\vec{u}}(P_0) = \nabla f(P_0)\cdot\vec{u}$.) But moving along K means that the constraints $g(x,y)=c$ remain satisfied. So if $\nabla f(P_0) \not\perp K$, we can increase $f(x,y)$ without violating constraints. Thus, P_0 is not a maximum. $\nabla f(P_0) \perp K$ and $\nabla g(P_0) \perp K$ means $\nabla f(P_0) \parallel \nabla g(P_0)$. We have then:

Th. Let $f(x,y)$, $g(x,y)$ be smooth, c be given. Consider the problem:

$$(*) \begin{cases} \text{Maximize (or Minimize)} : & f(x,y) \\ \text{Subject to} : & g(x,y)=c. \end{cases}$$

Suppose P_0 is a solution to $(*)$. Then either P_0 satisfies for some number λ:

$$\nabla f(P_0) = \lambda \nabla g(P_0), \quad g(P_0) = c$$

or P_0 is an endpoint of the constraint or $\nabla g(P_0) = \vec{0}$. To find P_0 compare values of $f(x,y)$ at all such points. λ is called the Lagrange multiplier.

This method works very well if we know that our problem has a solution. It is helpful to know that if the set of points $g(x,y)=c$ is closed and bounded, the $(*)$ has a solution.

Ex: Find the maximum and minimum values of $f(x,y)=x+y$ on the circle $g(x,y)=x^2+y^2=4$.

$$\begin{cases} \text{Maximize (Minimize)} & : \quad f(x,y)=x+y \\ \text{Subject to} & : \quad g(x,y)=x^2+y^2=4 \end{cases}$$

$\nabla f(x,y)=\vec{i}+\vec{j}$, $\nabla g(x,y)=2x\vec{i}+2y\vec{j}$.

At what points $P_0=(x_0,y_0)$ on the circle $x^2+y^2=4$ do we have

$$\nabla f(x_0,y_0) \parallel \nabla g(x_0,y_0) \ ?$$

$\vec{i}+\vec{j} \parallel 2x\vec{i}+2y\vec{j} \Leftrightarrow x=y$

$$\begin{cases} x_0^2+y_0^2=4 \\ x_0=y_0 \end{cases}$$

So $2x_0^2=4$, $x_0=\pm\sqrt{2}$, $x_0=y_0$. Thus we have two points on the circle when the gradients are parallel:

$$P_0=(\sqrt{2},\sqrt{2}) \text{ and } P_1=(-\sqrt{2},-\sqrt{2}).$$

No endpoints, no points on the circle where $\nabla g(x,y)=\vec{0}$.

The constrains curve has no endpoints, it constitutes a closed and bounded set, so our $f(x,y)$ has both extrema on the circle. They can only happen at P_0 or P_1. Comparing the values:

$f(P_0) = 2\sqrt{2}$ — maximum

$f(P_1) = -2\sqrt{2}$ — minimum

Our production problem:

$$\begin{cases} \text{Max}: \; x^{\frac{2}{3}} \cdot y^{\frac{1}{3}} \\ \text{Sub. to}: \; x+y = 3.78 \end{cases}$$

$$\nabla f = \frac{2}{3} x^{-\frac{1}{3}} y^{\frac{1}{3}} \vec{i} + \frac{1}{3} x^{\frac{2}{3}} y^{-\frac{2}{3}} \vec{j}$$

$$\nabla g = \vec{i} + \vec{j}$$

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

$$\begin{cases} \frac{2}{3} x^{-\frac{1}{3}} y^{\frac{1}{3}} = \lambda \\ \frac{1}{3} x^{\frac{2}{3}} \cdot y^{-\frac{2}{3}} = \lambda \end{cases}$$

$$\frac{2}{3} x^{-\frac{1}{3}} \cdot y^{\frac{1}{3}} = \frac{1}{3} x^{\frac{2}{3}} \cdot y^{-\frac{2}{3}}$$

$$\frac{2}{3} y = \frac{1}{3} x \quad , \quad 2y = x$$

So $2y + y = 3.78$. $y = 1.26$, $x = 2.52$

$$f(1.26, 2.52) \cong 2$$

With the budget \$3,780, we should use 1.26 units of y and 2.52 units of x.

For problems:

$$(**)\quad \begin{cases} \text{Maximize (or Minimize)} &: f(x,y) \\ \text{Subject to} &: g(x,y) \leq c. \end{cases}$$

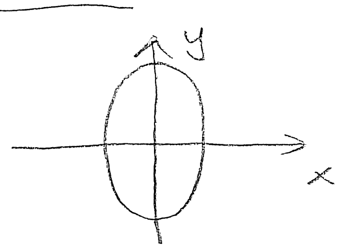
We use the same technique to find points on the boundary $g(x,y) = c$ and then compare the values at those points with values of $f(x,y)$ at all critical points in the region $g(x,y) < c$.

Ex: Find maximum and minimum of

$$f(x,y) = xy$$

subject to

$$g(x,y) = 4x^2 + y^2 = 8.$$



$$\nabla g(x,y) = 8x\vec{i} + 2y\vec{j}$$

$$\nabla f(x,y) = y\vec{i} + x\vec{j}$$

$\nabla g \parallel \nabla f$ at (x,y) on the ellipse if:

$$\begin{cases} \lambda 8x = y \\ \lambda 2y = x \\ 4x^2 + y^2 = 8 \end{cases} \quad *$$

$$2\lambda y^2 = 8\lambda x^2$$

If $\lambda = 0$, then $x = y = 0$, so constraints are violated.

So $\lambda \neq 0$ and

$$y^2 = 4x^2$$

Thus $\quad y = \pm 2x$, which gives:

$$4x^2 + 4x^2 = 8$$

$$x^2 = 1$$

We get critical points along the ellipse:

$$(1, 2), \quad (1, -2), \quad (-1, 2), \quad (-1, -2).$$

Max

$$f(1, 2) = 2 = f(-1, -2)$$

$$f(1, -2) = -2 = f(-1, 2).$$

Min