April 7, 9

17.1 – 17.2 Parametrized Curves, Motion

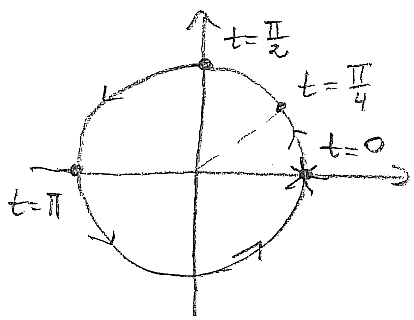
You are familiar with parametric representations of curves on the xy-plane.

Ex: What curve on the xy-plane is described by:

$$x = \cos t \quad , \quad y = \sin t \quad , \quad t \text{ in } [0, 2\pi] .$$

the parameter

Since $\cos^2 t + \sin^2 t = 1$, each point on the parametric curve is on the unit circle:

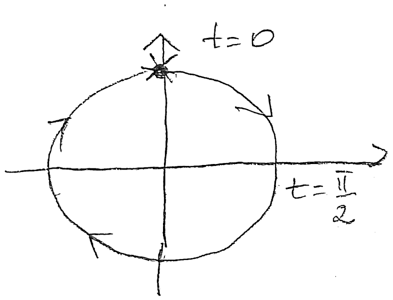


$t = 0 \quad \to \quad (1, 0)$

As t increases, we move counter-clockwise; at $t = 2\pi$, we are back at $(1, 0)$.

Of course, the unit circle has other parametric representations. For example:

$$x = \sin t \quad , \quad y = \cos t \quad , \quad t \in [0, 2\pi]$$



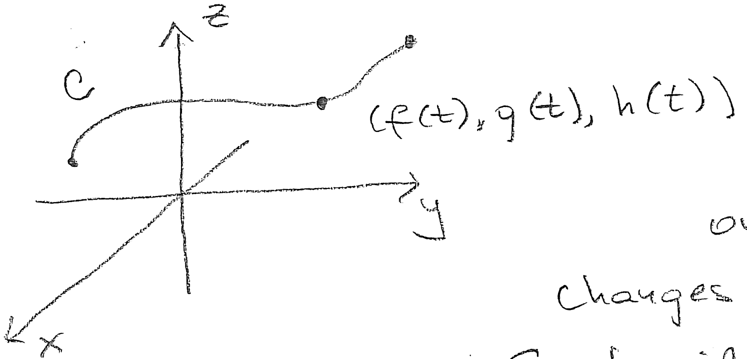
Same path, different parametrization.

Parametrizations provide a very convenient way of representing curves in the xyz-space.

A parametric curve in the xyz-space is a curve described by parametric equations:

$$x = f(t), \quad y = g(t), \quad z = h(t)$$

where the parameter t changes in an interval I.

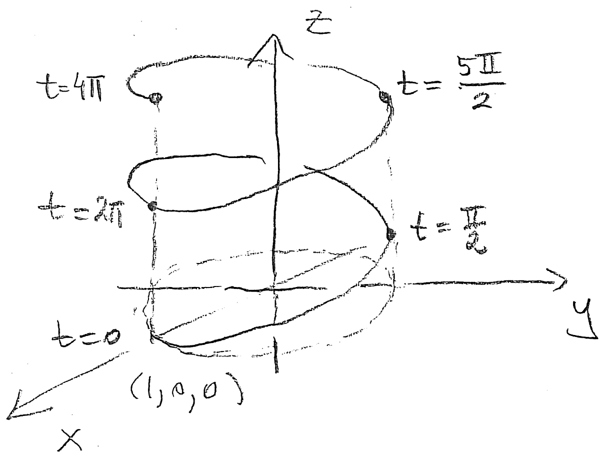


For each t we have a point $(f(t), g(t), h(t))$ on the curve C. As t changes, the point moves along C decribing a motion along the path C.

Ex: What curve is described by

$$x = \cos t, \quad y = \sin t, \quad z = t, \quad t \geq 0.$$

What motion along the curve is decribed by the parametrization?



Note that the projection onto the xy-plane:

$$x = \cos t, \quad y = \sin t$$

moves ccw about the unit circle. At the same time z increases so we move up.

A helix in the xyz-space.

The same helix can be parametrized by:

Ex: $x = \cos(2t)$, $y = \sin(2t)$, $z = 2t$, $t \geq 0$.

In the latter parametrization we traverse the helix twice as fast.

Parametrization in Vector Form

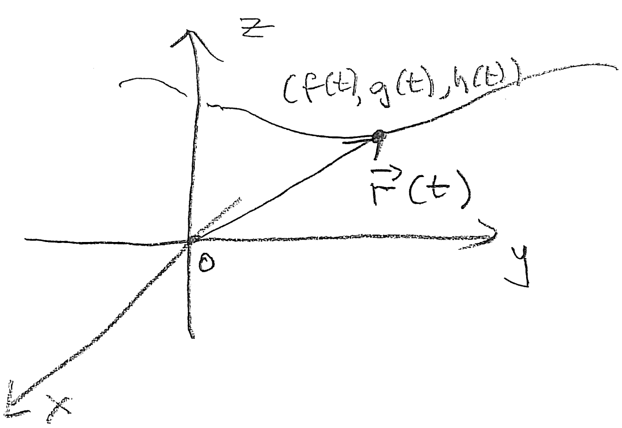
Let a parametrized curve

$$x = f(t), \quad y = g(t), \quad z = h(t), \quad t \text{ in } I$$

be given. We can write the parametrization as:

$$\vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}, \quad t \text{ in } I.$$

$\vec{r}(t)$ is called the position vector.



The position vector traces the curve as t changes.

The simplest way to describe a straight line in 3D is by a parametric representation.

Let L be the line passing through a point (x_0, y_0, z_0) and parallel to $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$. Then

$$\underline{L : x(t) = x_0 + tw_1, \quad y(t) = y_0 + tw_2, \quad z(t) = z_0 + tw_3,}$$

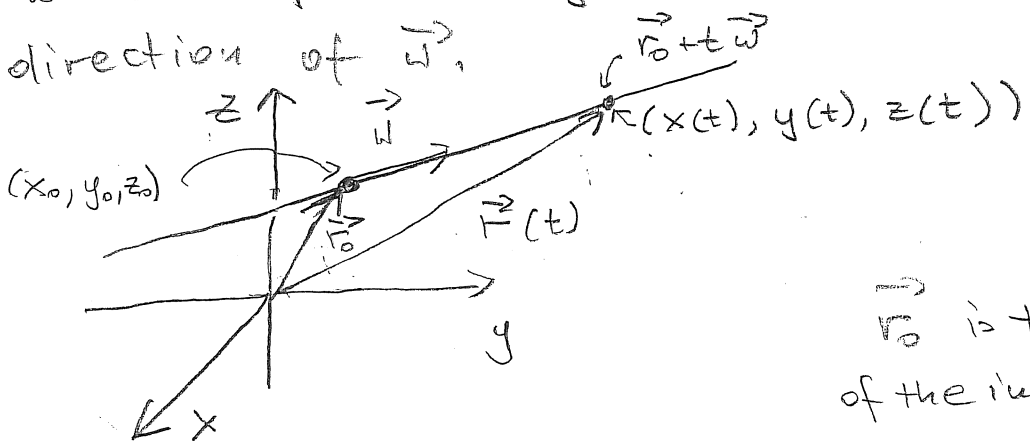
$$-\infty < t < +\infty.$$

In the vector form:

$$\underline{L: \vec{r}(t) = \vec{r_0} + t\vec{w}, \quad \vec{r_0} = x_0\vec{i} + y_0\vec{j} + z_0\vec{k},}$$

$$-\infty < t < +\infty$$

Of course, the displacement vector from (x_0, y_0, z_0) to $(x(t), y(t), z(t))$ is $tw_1\vec{i} + tw_2\vec{j} + tw_3\vec{k} \parallel \vec{w}$. So we start from (x_0, y_0, z_0) and move t units in the direction of $\vec{w}$.



$\vec{r_0}$ is the position vector of the initial point.

<u>Ex</u>: Let $P_0 = (2, -1, 3)$, $P_1 = (-1, 5, 4)$. Find a parametric representation of:

(a) The line, L, through P_0, P_1

(b) The segment, S, from P_0 to P_1.

(a) $\vec{w} = \overrightarrow{(2,-1,3)(-1,5,4)} = -3\vec{i} + 6\vec{j} + \vec{k}$ — the direction vector

$P_0 = (x_0, y_0, z_0) = (2, -1, 3)$ — initial point

$\vec{r}_0 = 2\vec{i} - \vec{j} + 3\vec{k}$ — the position vector of the initial point.

Parametric representation:

$$L: \vec{r}(t) = \vec{r}_0 + t\vec{w}, \quad -\infty < t < +\infty$$

Thus:

$$L: \vec{r}(t) = 2\vec{i} - \vec{j} + 3\vec{k} + t(-3)\vec{i} + t6\vec{j} + t\vec{k}$$

$$L: \vec{r}(t) = (2-3t)\vec{i} + (-1+6t)\vec{j} + (3+t)\vec{k}$$

In non-vector form:

$$L: \quad x(t) = 2-3t, \; y(t) = -1+6t, \; z(t) = 3+t$$

$$-\infty < t < +\infty$$

(b) Since $\vec{w}$ is $\overrightarrow{P_0P_1}$, in the representation

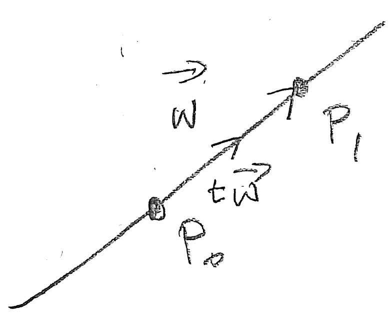
$$\vec{r}(t) = \vec{r}_0 + t\vec{w}$$

we are at P_0 for $t = 0$ and at P_1 at $t = 1$.

So:

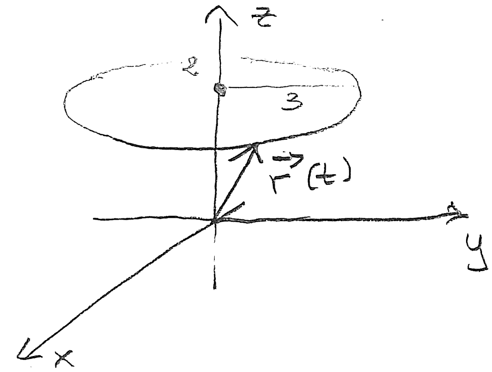
$$S: \vec{r}(t) = (2-3t)\vec{i} + (-1+6t)\vec{j} + (3+t)\vec{k}$$

$$0 \le t \le 1.$$



Note: A parametric representation gives us not only a path (L or S) but als a specific motion along the path.

Ex: Find a parametric equation of the circle of radius 3 parallel to the xy-plane centered at (0, 0, 2).



$x = 3\cos t, \quad y = 3\sin t, \quad z = 2$

$0 \le t \le 2\pi$

Or

$\vec{r}(t) = (3\cos t)\vec{i} + (3\sin t)\vec{j} + 2\vec{k}$

$0 \le t \le 2\pi.$

Of course, we use parametric representations to describe motion in xyz-space. The parameter t denotes time. What is the velocity and the acceleration?

Def: Let $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ be the position vector of an object at time t. The velocity vector, $\vec{v}(t)$, at time t is defined as:

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \vec{r}\,'(t) = \frac{d\vec{r}}{dt}.$$

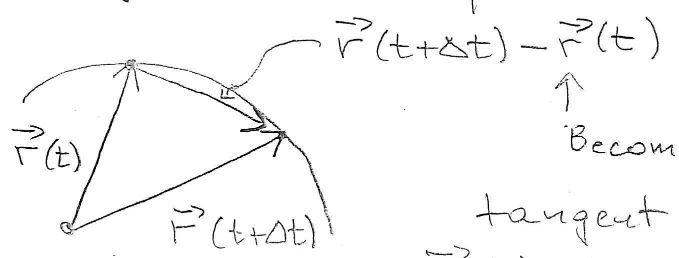
In terms of coordinates;

$$\vec{v}(t) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}.$$

Speed is defined as:

$$\text{Speed} = \|\vec{v}(t)\| = \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}.$$

$\vec{v}(t)$ is tangent to the path at each point;



Becomes more and more tangent as $\Delta t \to 0$.

$\vec{v}(t)$ points in the direction of the motion.

<u>Def</u>: Let $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ be the position vector of an object at time t. The acceleration vector $\vec{a}(t)$ is defined as

$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} = \frac{d\vec{v}}{dt} = \vec{v}\,'(t) = \vec{r}\,''(t) = \frac{d^2\vec{r}}{dt^2}.$$

In terms of coordinates:

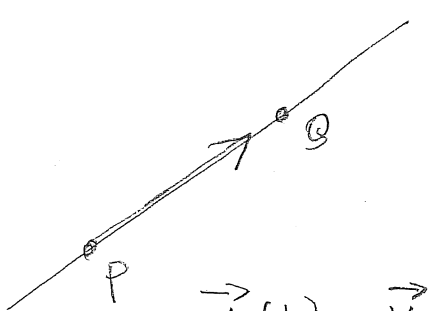
$$\vec{a}(t) = f''(t)\vec{i} + g''(t)\vec{j} + h''(t)\vec{k}.$$

$\vec{a}(t)$ reflects the changes in both direction and magnitude of the velocity $\vec{v}(t)$.

<u>Ex</u> A particle starts at the point $P = (3,2,-5)$ and moves along a straight line toward $Q = (5,7,-2)$ at a speed of $5\ \frac{cm}{sec}$. Let x, y, z be measured in cm.

(a) Find the particle's velocity vector.

(b) Find parametric equations for the particle's motion.



$\vec{v}(t) \parallel \overrightarrow{PQ} = 2\vec{i} + 5\vec{j} + 3\vec{k}$

t in seconds.

$\|\vec{v}(t)\| = 5\ \frac{cm}{sec}$.

$\vec{v}(t) \equiv \vec{v} = c\,2\vec{i} + c\,5\vec{j} + c\,3\vec{k}$ $\qquad c = ?$

$$\|\vec{v}\| = \sqrt{4c^2 + 25c^2 + 9c^2} = c\sqrt{38} = 5 \rightarrow c = \frac{5}{\sqrt{38}}$$

$$\vec{v}(t) \equiv \frac{5}{\sqrt{38}} \cdot \vec{PQ} = \frac{10}{\sqrt{38}}\vec{i} + \frac{25}{\sqrt{38}}\vec{j} + \frac{15}{\sqrt{38}}\vec{k}$$

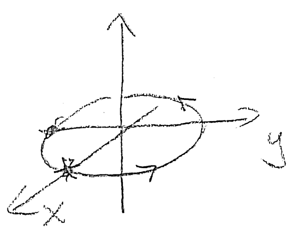
$$\vec{a}(t) = \vec{0}.$$

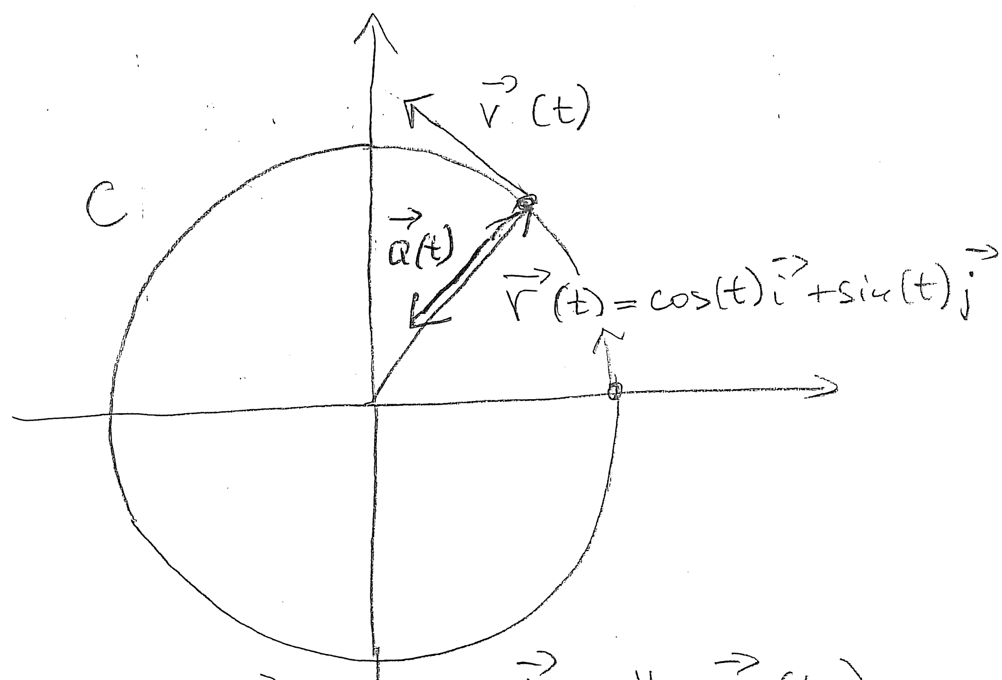
Indeed, neithe the direction nor the magnitude of the velocity changes.

<u>Ex</u>: Let $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j}$. Find $\vec{v}(t)$, $\vec{a}(t)$; $t \in [0, 2\pi]$

$x = \cos t$, $y = \sin t$, $z = 0$

$$\vec{v}(t) = -\sin(t)\vec{i} + \cos(t)\vec{j}$$





$$\vec{a}(t) = -\cos(t)\vec{i} - \sin(t)\vec{j} \quad \| \quad \vec{r}(t)$$

$$\vec{a}(t) \perp \vec{v}(t)$$

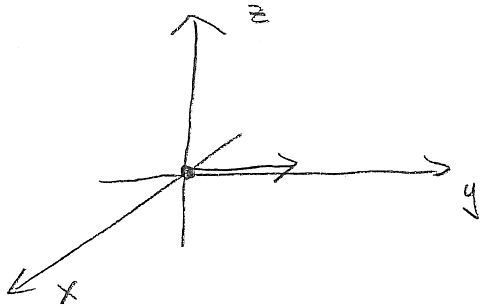
$\|\vec{v}(t)\| \equiv 1$. $\vec{a}(t)$ reflects changes in the direction of $\vec{v}(t)$.

$$\text{Length of } C = \int_0^{2\pi} \|\vec{v}(t)\| dt = \int_0^{2\pi} (\sin^2(t) + \cos^2(t))\, dt = 2\pi.$$

Ex : Consider the motion of an object:

$$\vec{r}(t) = t^2\vec{j}, \quad t \geq 0$$

Find $\vec{v}(t)$ and $\vec{a}(t)$. Describe the motion.



The object moves faster and faster along the y axis.

$\vec{v}(t) = 2t\vec{j}$ - parallel to the y-axis

$s(t) = \sqrt{4t^2} = 2t$ - speed increasing

$\vec{a}(t) = 2\vec{j}$ - acceleration not $\vec{0}$ as the magnitude of $\vec{v}(t)$ changes.

$\vec{a}(t)$ reflects changes in the magnitude of $\vec{v}(t)$.

Let $\vec{v}(t)$ be the velocity in a motion. Then

$$\text{Distance traveled between } t=a \text{ and } t=b = \int_a^b \|\vec{v}(t)\|\,dt$$

Suppose an object moves along a curve C, covers the curve once for t in $[a, b]$, then the length of C

$$\text{Length of } C = \int_a^b \|\vec{v}(t)\|\,dt$$

17.3 – 17.4 Vector Fields

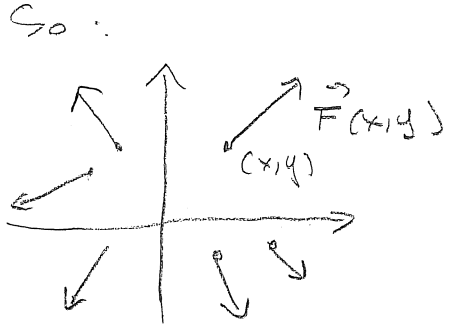
A vector field on the plane is a function which to each point (x,y) in some region prescribes a vector:

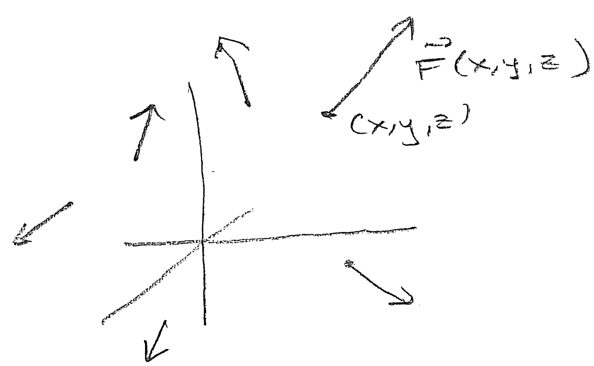
$$(x,y) \longrightarrow \vec{F}(x,y)$$

In 3D, it is a function which to each (x,y,z) prescribes a 3D vector $\vec{F}(x,y,z)$:

$$(x,y,z) \longrightarrow \vec{F}(x,y,z).$$

So:

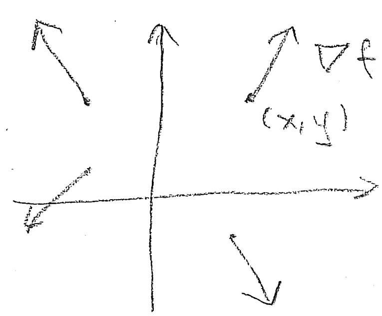




When we sketch vector fields, we usually scale length.
Vector fields are very important: force fields, current velocity fields etc.
We already know vector fields: if we have a function $f(x,y)$, then we have its <u>gradient vector field</u>:

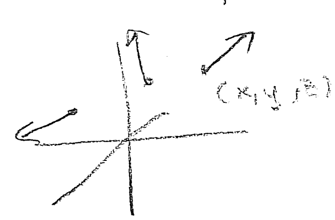
Given $f(x,y)$, we have

$$(x,y) \longrightarrow \nabla f(x,y)$$

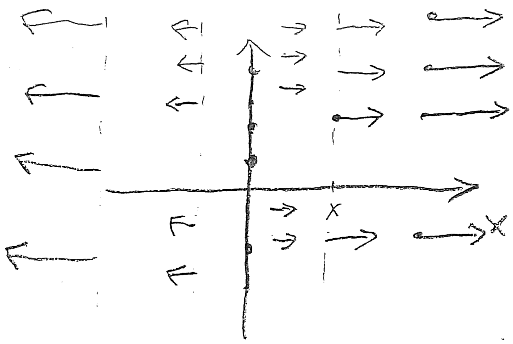


Given $f(x,y,z)$, we have

$$(x,y,z) \longrightarrow \nabla f(x,y,z)$$

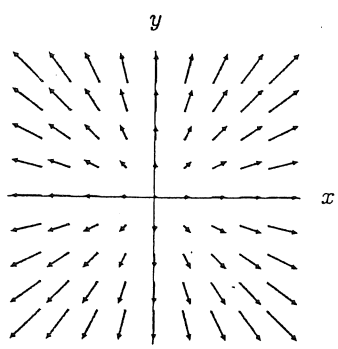


Ex : $f(x,y) = x^2$. Sketch the gradient vector field for f.



$\nabla f(x,y) = 2x\vec{i}$

Ex.: Sketch the vector field $\vec{F}(x,y) = 2x\vec{i} + 2y\vec{j}$.



Observe that $\vec{F}(x,y) = \text{grad } f(x,y)$, where $f(x,y) = x^2 + y^2$.

Denote $\vec{r} = x\vec{i} + y\vec{j}$. Then $\vec{F}(x,y)$ can be written as
$$\vec{F}(\vec{r}) = 2\vec{r}.$$

$\vec{F}(x,y) \parallel x\vec{i} + y\vec{j} = \vec{r}$ the position vector of (x,y)
$\vec{F}(x,y)$ at each point (x,y) point directly away from the origin. The magnitude increases as we move away from the origin. If possible, we write vector fields in terms of $\vec{r} = x\vec{i} + y\vec{j}$.

Def: A vector field $\vec{F}(x,y)$ or ($\vec{F}(x,y,z)$) is called a gradient vector field if for some $f(x,y)$ ($f(x,y,z)$):
$$\vec{F}(x,y) = \nabla f(x,y) \qquad (\vec{F}(x,y,z) = \nabla f(x,y,z)).$$

Perhaps every vector field is a gradient field?

Ex: $\vec{F}(x,y) = 2xy\,\vec{i} + xy\,\vec{j}$

Suppose that for some $f(x,y)$

$$\vec{F}(x,y) = \nabla f(x,y) = f_x(x,y)\,\vec{i} + f_y(x,y)\,\vec{j}.$$

Then $f(x,y)$ has to be such that

$$f_x(x,y) = 2xy, \quad f_y(x,y) = xy$$

?

$$f(x,y) = x^2y + C(y)$$

Doesn't seem possible.

$$f_y(x,y) = x^2 + C'(y) \quad \text{not } xy$$

There is a simple way to test if a given vector field

$$\vec{F}(x,y) = F_1(x,y)\,\vec{i} + F_2(x,y)\,\vec{j}$$

is a gradient field. If it is, then for some $f(x,y)$:

$$F_1(x,y) = f_x(x,y), \quad F_2(x,y) = f_y(x,y).$$

As $f_{xy}(x,y) = f_{yx}(x,y)$, we have then:

$$\frac{\partial F_1}{\partial y} = f_{xy} = f_{yx} = \frac{\partial F_2}{\partial x}.$$

So:

If $\dfrac{\partial F_1}{\partial y} \neq \dfrac{\partial F_2}{\partial x}$, then $\vec{F}$ is not a gradient field.

This is very important!

For $\vec{F}(x,y) = 2xy\vec{i} + xy\vec{j}$, we have:

$$\frac{\partial}{\partial y}[2xy] = 2x, \quad \frac{\partial}{\partial x}[xy] = y$$

So $\frac{\partial F_1}{\partial y} \neq \frac{\partial F_2}{\partial x}$. Not a gradient field.