MTH 436 **Final Exam - Tips** **Spring 2015**

Our Final Exam is scheduled for Monday, May 4, 3 pm - 6 pm, Lippitt 204. The exam is comprehensive and covers all classes and all homework assignments of this semester.

As always, you are expected to know and be able to state all definitions, theorems and propositions given in class. Basically, you are expected to know all proofs done in class with a special emphasis on those listed below. You should be familiar with all examples presented in class. You should know solutions to all homework problems. On the exam, you will have to state some definitions, theorems and propositions given in class as well as prove things. Some proofs will come from those given in class. Some exam problems will be similar to homework problems. In addition, some problems not done in class and not similar to homework problems will appear. Below are a few questions of the kind you should expect.

Analysis

- State the definition of a Lipschitzian function.

- Prove if f is Lipschitzian on an interval I, then f is uniformly continuous on I (Prop 41.3).

- Prove if f is differentiable on an open interval I and f' is bounded in I, then f is Lipschitzian on I (Prop 42.1).

- Prove that if f is Lipschitzian and differentiable in an open interval I, then f' is bounded in I. (Prop 42.2).

- Give and example of a function f that is uniformly continuous in $[a, b]$ but not Lipschitzian.

- State the definition of the Riemann integral $\int_a^b f$. (Def 44.4). Show from the definition that the Dirichlet function is not Riemann integrable in $[0, 1]$. (Ex 44.1)

- Show from the definition of the integral that a constant function, $f(x) = c$ for all $x \in [a, b]$, is integrable on $[a, b]$ and $\int_a^b f = c(b - a)$. (Ex 44.2)

- Prove (a) or (b) or (c) of Th. 45.1.

- State the Cauchy Condition for Integrability. (Th 46.1). Prove that if f is integrable on $[a, b]$, then f satisfies the Cauchy Condition for Integrability. (The first part of the proof of Th 46.1.)

- Define the oscillation $\omega(f, [a, b])$ of a bounded function f on $[a, b]$.

Remark: On the exam you may need Prop 47.2 which says: that
$\omega(f, [a, b]) = \sup\{|f(x) - f(y)| : x, y \in [a, b]\}$. You can assume this proposition as given. You will not be expected to prove the proposition.

- State the theorem (Th 46.2) which gives a necessary and sufficient condition for integrability in terms of oscillations.

- Prove that if f is continuous in $[a, b]$, then f is integrable in $[a, b]$. (Th 47.1).

- Prove that $\omega(|f|, [a, b]) \leq \omega(f, [a, b])$. Use that fact to prove that if f is integrable, then $|f|$ is integrable on $[a, b]$. Give an example that $|f|$ is integrable on $[a, b]$ but f is not.

- State the First Fundamental Theorem of Calculus (Th. 50.1).

- State the Second Fundamental Theorem of Calculus (Th. 51.1).

- Prove (a) of Th. 51.1; that is, prove that the running integral of f appearing there is Lipschitzian.

- State the definition of pointwise convergence of a sequence of functions f_n to a function f on an interval I.

- Be familiar with all examples of sequences of functions done in class. You may need them.

- State the definition of uniform convergence of a sequence of functions f_n to a function f on an interval I.

- Give an example of a sequence of continuous functions f_n that converges on an interval I to a function f which is not continuous on I. Show that the convergence is not uniform.

- Give an example of a sequence of continuous functions $\{f_n\}_{n=1}^{+\infty}$ that converges on an interval $[a, b]$ to a function f but the integrals $\int_a^b f_n$ do not converge to the integral of the limit $\int_a^b f$. Show that the convergence is not uniform.

- Let $\{f_n\}_{n=1}^{+\infty}$ be a sequence of integrable functions on $[a, b]$ that converges uniformly on $[a, b]$ to an integrable function f. Prove that the integrals $\int_a^b f_n$ converge to the integral of the limit $\int_a^b f$. (Prop. 53.1).

- State the Necessary Condition for Convergence for a Series of Constants (Th. 57.3). Prove the theorem.

- State the Cauchy Condition for a Series of Constants (Th. 57.4).

- Prove that the harmonic series diverges (Th. 57.5).

- Prove Prop. 58.4.

- State and prove the Standard Comparison Test (Th 58.1).

- State and prove the Limit Comparison Test (Th 59.1).

- State the Leibnitz Theorem. Show that the alternating harmonic series converges.

- Prove that a series that is convergent absolutely is convergent. Give an example of a series that converges conditionally.

- State the definition of a series of functions that converges (a) pointwise (b) uniformly.

- Prove that if a series is convergent uniformly on $[a, b]$, the series of integrals converges to the integral of the sum. (Th. 61.4).

- State and prove the Weierstrass M-Test (Th. 62.2). Know how to use it in examples.

- State the definition of a power series.

- State the theorem about convergence of a power series (Th. 62.3).

- State the theorem about differentiabilty of the sum of a power series $f(x) = \sum_{n=0}^{+\infty} a_n(x - x_0)^n$ (Th. 64.1).

- Let $r > 0$. Let $f(x) = \sum_{n=0}^{+\infty} a_n(x - x_0)^n$ for all $x \in (x_0 - r, x_0 + r)$. What is the formula for a_n?

- State the definition of the Taylor series of $f(x)$ at x_0. State the Taylor Theorem (Th 65.1). Use the theorem to prove that $e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$.

Particularly suitable homework problems for **Analysis** part:

Homework 2: Problems 5, 6
Homework 3: Problems 1, 2, 3, 5, 6
Homework 4: All problems
Homework 5: Problems 1, 2, 5, 6, 7, 10, 12

From Exam 1, be sure to review Problems 2, 5.

Topology

- Prove Prop 39.1.

- State Th 39.1 (The Cantor Theorem). Explain how the Nested Intervals Theorem is a special case of Th. 39.1.

- State the definition of a dense subset of a metric space. Prove Prop 41.1.

- State the definition of a separable metric space.

- Prove that $\mathbf{R^2}$ is separable.

- Give an example of a metric space that is not separable. (Ex 41.4)

- State the definition of a topological space. (Def 67.1)

- Let X be a given set. Assume that X is not countable. Define a family of open sets $\mathcal{O}$ in X as follows:
  (i) $\emptyset \in \mathcal{O}$
  (ii) For $A \subseteq X$, $A \in \mathcal{O}$ if and only if the complement $X - A$ is countable.
Prove that $(X, \mathcal{O})$ is a topological space.
$\mathcal{O}$ is called the cocountable topology in X. Is the cocountable topology metrizable?

- State the definition of
  (a) a compact metric space;
  (b) a sequentially compact metric space;
For a metric space, is compactness equivalent to sequential compactness? State without proof.

- Prove that the closed unit ball in $(C([0,1]), d_{sup})$ is not sequentially compact. (Ex 69.1)

- Let (X, d), (Y, ρ) be metric spaces. State the definition of a continuous mapping $f : X \to Y$. (Def 70.1)

- Let (X, d), (Y, ρ) be metric spaces. State the Cauchy characterization of continuity of a mapping $f : X \to Y$. (Th 70.1)

- Let (X, d), (Y, ρ) be metric spaces. Assume that a mapping $f : X \to Y$ is continuous. Prove that for every open set $V \subseteq Y$ the preimage $f^{-1}(V)$ is open in X. (Th 70.2)

- Let (X, d), (Y, ρ) be metric spaces. State the definition of a uniformly continuous mapping $f : X \to Y$. (Def 70.2)

- Let (X, d), (Y, ρ) be metric spaces. Assume (X, d) is compact. Assume that a mapping $f : X \to Y$ is continuous. Prove that f is uniformly continuous. (Th 70.3)

- Let (X, d), (Y, ρ) be metric spaces. Assume (X, d) is compact. Assume that a mapping $f : X \to Y$ is continuous. Prove that the image $f(X)$ is compact. (Th 71.1)

Particularly suitable homework problems for **Topology** part:

Homework 1: Problems 1, 4
Homework 2: Problems 1, 2
Homework 6: All problems

Happy studying! I hope you all get As!