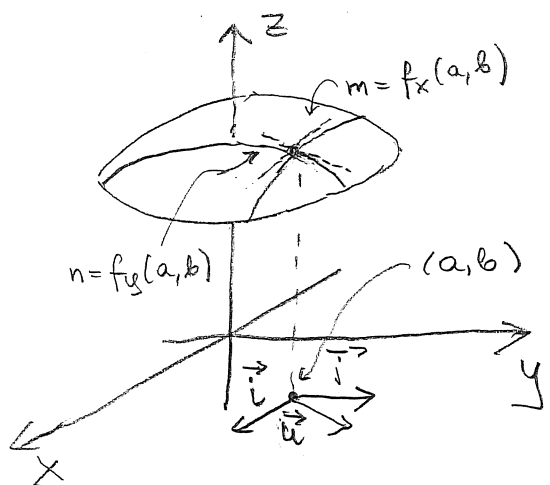


14.4 Gradients . Directional Derivatives

Let $z = f(x, y)$ be given. The partial derivatives $f_x(a, b)$, $f_y(a, b)$ at some point (a, b) give the rates of change in the direction of x and the direction of y :



$f_x(a, b)$ gives us the rate of change when we walk from (a, b) in the direction of $\vec{i}$, $f_y(a, b)$ if we walk in the direction of $\vec{j}$.

What is the rate of change if we move from (a, b) in the direction of some other unit vector $\vec{u}$?

This rate of change is called the directional derivative of $f(x, y)$ at (a, b) in the direction of $\vec{u}$.

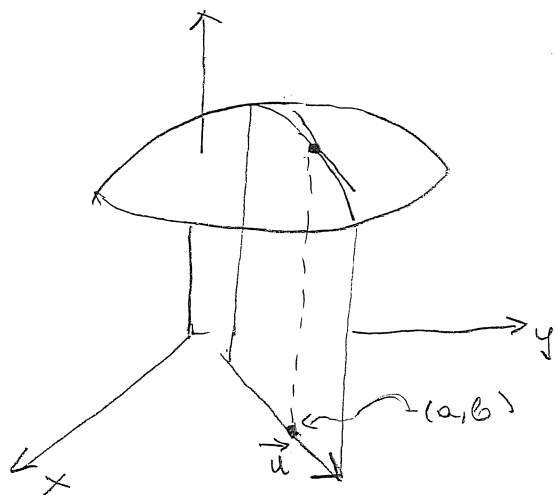
Def: Let $\vec{u} = u_1\vec{i} + u_2\vec{j}$, $\|\vec{u}\| = 1$, $z = f(x, y)$, (a, b) be given. The directional derivative of $f(x, y)$ at (a, b) in the direction of $\vec{u}$ is:

$$f_{\vec{u}}(a, b) = \lim_{h \rightarrow 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$

$$(a, b) \rightarrow (a + hu_1, b + hu_2)$$

Note: The definition talks about both-sided limit "lim". Thus h can be positive or negative.

The limit exists if the cross-section of the graph by the vertical plane containing $\vec{u}$ and $(a, b, f(a, b))$ has the tangent line at $(a, b, f(a, b))$:



Thus $f_{\vec{u}}(a, b)$ also tells us that if we go in the opposite direction, $-\vec{u}$, $f(x, y)$ will be changing at the rate $-f_{\vec{u}}(a, b) = f_{-\vec{u}}(a, b)$

Indeed:

$$\begin{aligned} f_{-\vec{u}}(a, b) &= \lim_{h \rightarrow 0} \frac{f(a + h(-u_1), b + h(-u_2)) - f(a, b)}{h} = \\ &= \lim_{h \rightarrow 0} \frac{f(a + (-h)u_1, b + (-h)u_2) - f(a, b)}{-h} = \\ &= -f_{\vec{u}}(a, b). \end{aligned}$$

Note, the displacement vector from (a, b) to $(a + hu_1, b + hu_2)$ is $\vec{d} = (hu_1)\vec{i} + (hu_2)\vec{j} = h \cdot \vec{u}$. So $\|\vec{d}\| = h$ (or more precisely $|h|$).

This is why we need $\vec{u}$ to be the unit vector so the magnitude of h is the magnitude of the displacement. Geometrically, $f_{\vec{u}}(a, b)$ is the slope of the cross-section with the vertical plane parallel to $\vec{u}$ through $(a, b, f(a, b))$. Clearly:

$$f_x(a, b) = f_{\vec{i}}(a, b), \quad f_y(a, b) = f_{\vec{j}}(a, b).$$

Def: Let $f(x, y)$, (a, b) , and a vector $\vec{v}$ be given. Then

$$f_{\vec{v}}(a, b) = f \frac{\vec{v}}{\|\vec{v}\|}(a, b).$$

How do we calculate directional derivatives?

Using the so-called gradient vector.

Def: The gradient vector of $f(x, y)$ at (a, b) is defined as

$$\text{grad } f(a, b) = f_x(a, b)\vec{i} + f_y(a, b)\vec{j}.$$

In general:

$$\text{grad } f(x, y) = f_x(x, y) \vec{i} + f_y(x, y) \vec{j}$$

or

$$\text{grad } f = f_x \vec{i} + f_y \vec{j}$$

Ex: Let $f(x, y) = x^3 y + 3y + x$. Find $\text{grad } f(x, y)$. Find $\text{grad } f(1, 0)$.

$$f_x = 3x^2 y + 1, \quad f_y = x^3 + 3$$

$$\text{grad } f = (3x^2 y + 1) \vec{i} + (x^3 + 3) \vec{j}$$

$$\underline{\text{grad } f(1, 0) = \vec{i} + 4\vec{j}}$$

$\text{grad } f$ is a vector!

Th: Let $f(x, y)$ be differentiable at (a, b) , $\vec{u}$ be the unit vector. Then

$$\begin{aligned} f_{\vec{u}}(a, b) &= \text{grad } f(a, b) \cdot \vec{u} = \\ &= f_x(a, b) u_1 + f_y(a, b) u_2. \end{aligned}$$

Gradient is also denoted:

$$\underline{\text{grad } f = \nabla f}$$

Ex: Let $\vec{u} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$, $f(x, y) = x^2 + y^2$.

Find $f_{\vec{u}}(1, 0)$.

We use gradient.

$$\text{grad } f(x, y) = 2x\vec{i} + 2y\vec{j}$$

$$\text{grad } f(1, 0) = 2\vec{i}$$

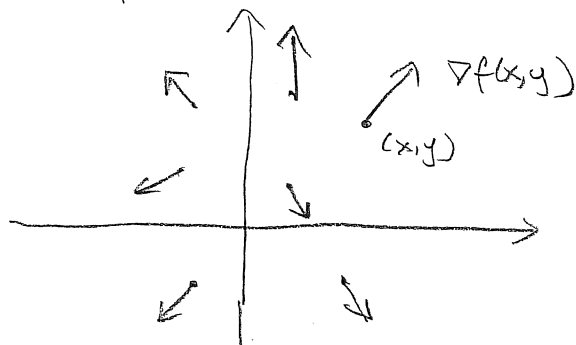
$$f_{\vec{u}}(1, 0) = 2\vec{i} \cdot \vec{u} = 2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}.$$

So 'finding' directional derivatives is as easy as finding gradients.

Let $f(x, y)$, differentiable, be given. Then we have the gradient vector at each point:

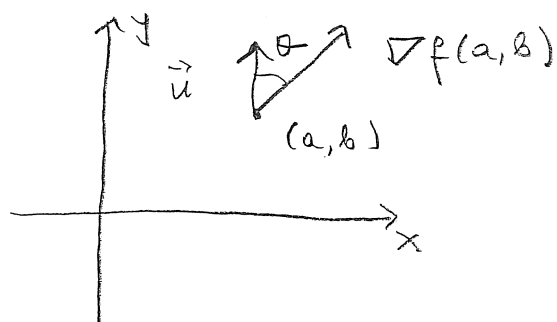
$$\nabla f(x, y)$$

So we have a vector field - the gradient field on the xy -plane:



Gradient Geometrically

Let's have $f(x, y)$, (a, b) , $\nabla f(a, b)$:



How about directional derivatives at (a, b) ?

Let $\vec{u}$ be a unit vector.

$$f_{\vec{u}}(a, b) = \nabla f(a, b) \cdot \vec{u} = \|\nabla f(a, b)\| \cdot \|\vec{u}\| \cos(\theta)$$

$f_{\vec{u}}(a, b)$ is the largest if $\cos(\theta) = 1$; that is, when $\theta = 0$, so $\vec{u}$ is parallel to $\nabla f(a, b)$.

Thus:

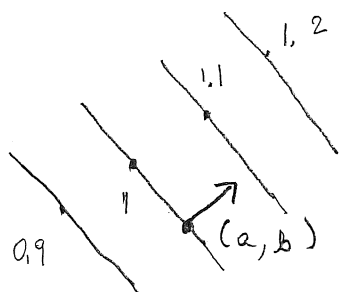
Th: Let $f(x, y)$ be differentiable at (a, b) , $\nabla f(a, b) \neq \vec{0}$.

Then:

- (1) $\nabla f(x, y)$ points in the direction of the largest rate of change of f at (a, b) . (The direction of the fastest growth.)
- (2) This largest rate of change is $\|\nabla f(a, b)\|$.
- (3) $\nabla f(a, b)$ is perpendicular to the contour line of $f(x, y)$ through (a, b) .

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Why (3)? Since $f(x,y)$ is differentiable at (a,b) , locally near (a,b) the graph of $z=f(x,y)$ is flat. Thus, locally contour lines are parallel:



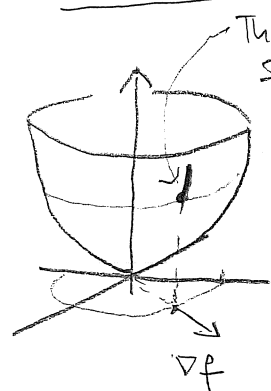
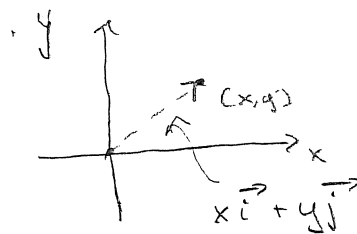
You obtain fastest growth by moving perpendicularly to contours in the direction of increasing values. So $\nabla f(a,b)$

that points toward fastest increase must be perpendicular to contours.

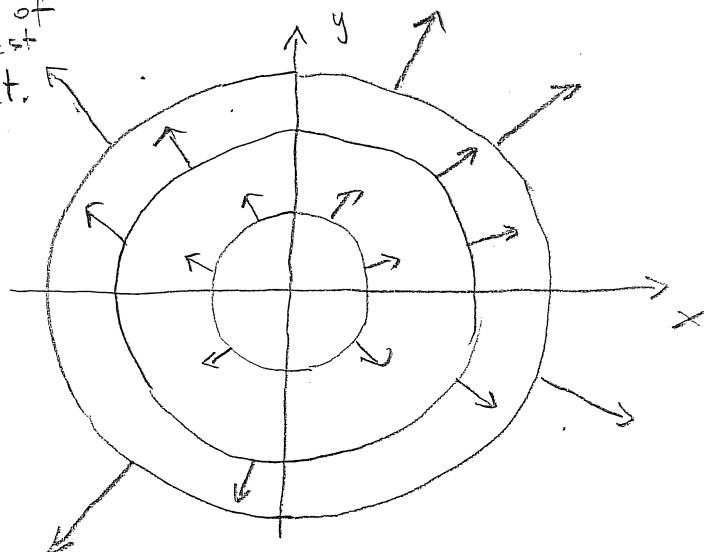
Ex: Let $f(x,y) = x^2 + y^2$. Sketch the contour diagram and the gradient field in one coordinate system.

$$\nabla f(x,y) = 2x\vec{i} + 2y\vec{j}$$

parallel the vector from $(0,0)$ to (x,y) .



The path of steepest ascent.



Magnitudes of vectors are scaled.

Ex A square metal plate is placed in the xy -plane in such a way that $0 \leq x \leq 3$, $0 \leq y \leq 3$. ^(All measurements in meters.) The temperature at each point (x, y) of the plane is given by:

$$T(x, y) = \frac{100}{x^2 + y^2 + 1} \text{ in } ^\circ\text{F.}$$

(a) Find the direction of the greatest increase in temp. at $(1, 2)$.
What is the greatest rate of increase at the point $(1, 2)$?

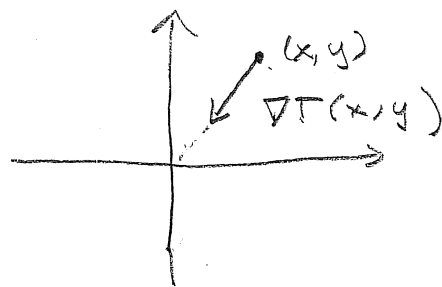
(b) Find the rate of increase at $(1, 2)$ in the direction $\vec{u} = \frac{2}{\sqrt{5}}\vec{i} + \frac{1}{\sqrt{5}}\vec{j}$.

Clearly, we have to find the gradient $\nabla T(x, y)$ first.

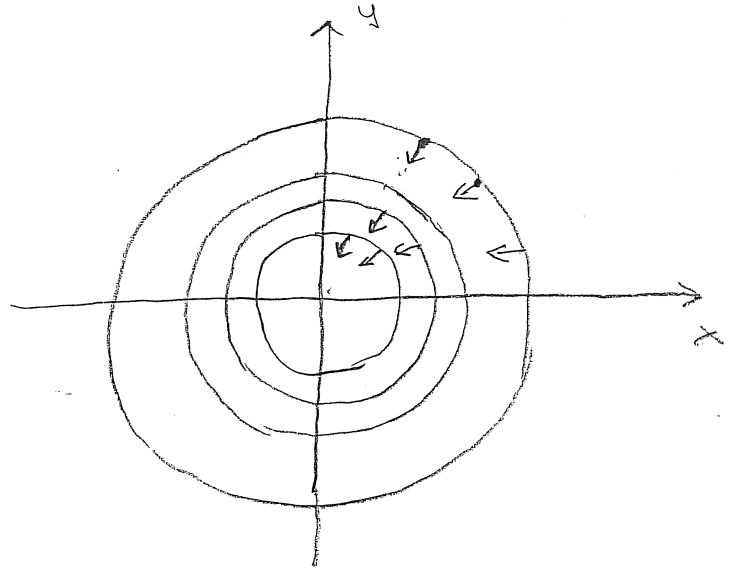
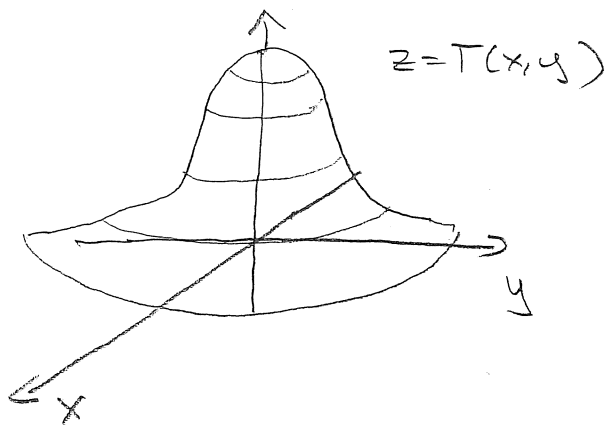
$$\begin{aligned} T_x(x, y) &= \frac{\partial}{\partial x} \left[\frac{100}{x^2 + y^2 + 1} \right] = \frac{\partial}{\partial x} \left[100 \underbrace{(x^2 + y^2 + 1)^{-1}}_{\text{const}} \right] = \\ &= 100 \cdot -(x^2 + y^2 + 1)^{-2} \cdot 2x = \\ &= - \frac{100}{(x^2 + y^2 + 1)^2} \cdot 2x = T_x(x, y) \end{aligned}$$

$$T_y(x, y) = - \frac{100}{(x^2 + y^2 + 1)^2} \cdot 2y$$

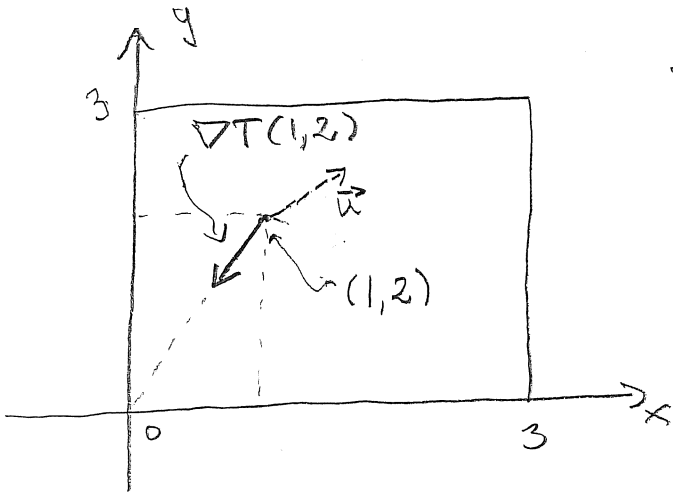
$$\nabla T(x, y) = - \frac{100}{(x^2 + y^2 + 1)^2} (2x\vec{i} + 2y\vec{j})$$



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Contours are concentric circles centered at $(0,0)$.
At each point the gradient $\nabla T(x,y)$ points toward the origin.



$$\nabla T(1,2) = -\frac{200}{36} \vec{i} - \frac{400}{36} \vec{j}$$

↑

The direction of the greatest increase in temperature at $(1,2)$. (Toward the origin). This answers (a).

$$(b) \quad \|\nabla T(1,2)\| = \sqrt{\left(\frac{200}{36}\right)^2 + \left(\frac{400}{36}\right)^2} \approx 77.16 \frac{^\circ\text{F}}{\text{m}}$$

$$(c) \quad T_{\vec{u}}(1,2) = \nabla T(1,2) \cdot \vec{u} =$$

$$= -\frac{200}{36} \cdot \frac{2}{\sqrt{5}} - \frac{400}{36} \cdot \frac{1}{\sqrt{5}} \approx -9.93 \frac{^\circ\text{F}}{\text{m}}$$

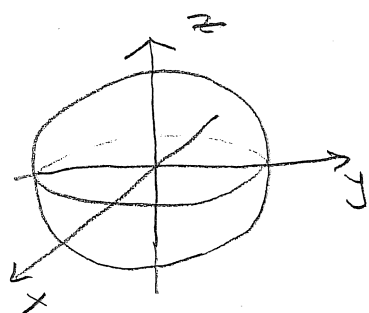
14.5 Gradients of Functions of Three Variables

Ex: $f(x, y, z) = x^2 + y^2 + z^2$

We cannot "graph" functions of three variables.

We can graph their level surface for a given k :

$$x^2 + y^2 + z^2 = k.$$



We can, of course, define three partial derivatives at each point (a, b, c) :

$$f_x(a, b, c), f_y(a, b, c), f_z(a, b, c)$$

by fixing two variables and differentiating with respect to the third. For example:

$$f_z(a, b, c) = \frac{d}{dz} \Big|_{z=c} [f(a, b, z)].$$

In our example:

$$f_x(x, y, z) = 2x, f_y(x, y, z) = 2y, f_z(x, y, z) = 2z.$$

We can define the directional derivative of

a function of three variables $f(x, y, z)$
in the direction of $\vec{u} = u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k}$,

$$\|\vec{u}\| = 1:$$

$$f_{\vec{u}}(a, b, c) = \lim_{h \rightarrow 0} \frac{f(a+hu_1, b+hu_2, c+hu_3) - f(a, b, c)}{h}.$$

This is the rate of change in the direction of $\vec{u}$.

We can define the gradient vector:

$$\nabla f(x, y, z) = f_x(x, y, z)\vec{i} + f_y(x, y, z)\vec{j} + f_z(x, y, z)\vec{k}.$$

In our example, $f(x, y, z) = x^2 + y^2 + z^2$,

$$\nabla f = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}.$$

As before, if $f(x, y, z)$ is differentiable at (a, b, c) :

$$f_{\vec{u}}(a, b, c) = \nabla f(a, b, c) \cdot \vec{u} \quad \frac{\text{units of } f}{\text{units of dist in dir } \vec{u}}$$

As before:

If $\nabla f(a, b, c) \neq \vec{0}$, then

- $\nabla f(a, b, c)$ points in the direction of the greatest rate of change of f at (a, b, c) .
- $\|\nabla f(a, b, c)\|$ is this greatest rate of change
- $\nabla f(a, b, c)$ is perpendicular to the level surface through (a, b, c) .

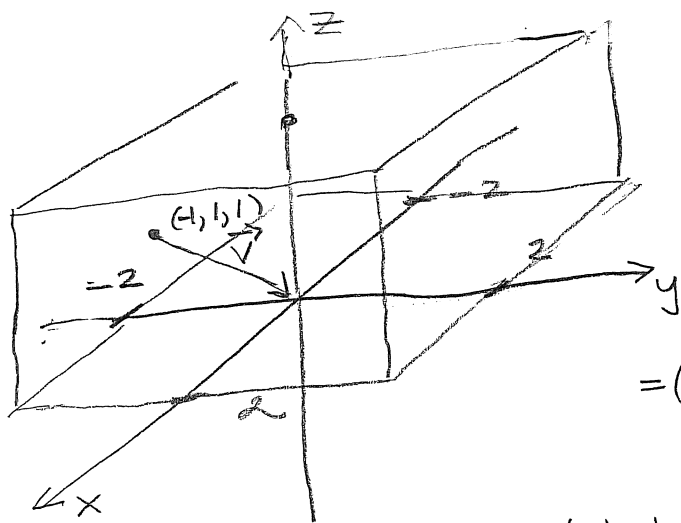
Ex Suppose that the function $F(x, y, z) = x^2 + y^4 + x^2 z^2$ gives concentration of salt, in gr/gal, at any point (x, y, z) of a rectangular tank of water occupying the region

$$-2 \leq x \leq 2, \quad -2 \leq y \leq 2, \quad 0 \leq z \leq 2.$$

(All measurements in meters.) Suppose you are at the point $(-1, 1, 1)$.

(a) In what direction should you move if you want the concentration to increase the fastest?

(b) If you move from $(-1, 1, 1)$ toward the origin $(0, 0, 0)$, how fast is the concentration changing?



(a)

$$\nabla F(x, y, z) =$$

$$= (2x + 2xz^2)\vec{i} + 4y^3\vec{j} + 2x^2z\vec{k}$$

$$\nabla F(-1, 1, 1) = -4\vec{i} + 4\vec{j} + 2\vec{k}$$

↑ The direction of greatest increase in concentration.

$$\|\nabla F(-1, 1, 1)\| = \sqrt{16 + 16 + 4} = 6 \frac{\text{g/gal}}{\text{m}}$$

$$(b) \quad \vec{v} = \overrightarrow{(-1, 1, 1)(0, 0, 0)} = \vec{i} - \vec{j} - \vec{k}, \quad \|\vec{v}\| = \sqrt{3}$$

$$\vec{u} = \frac{1}{\sqrt{3}} \vec{i} - \frac{1}{\sqrt{3}} \vec{j} - \frac{1}{\sqrt{3}} \vec{k}$$

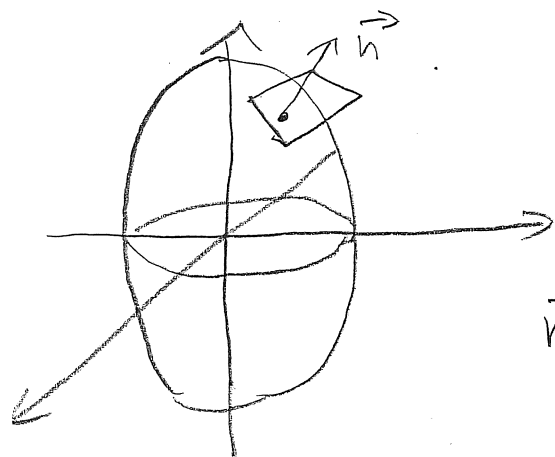
$$F_{\vec{v}}(-1, 1, 1) = F_{\vec{u}}(-1, 1, 1) = \nabla F(-1, 1, 1) \cdot \vec{u} =$$

$$= -\frac{4}{\sqrt{3}} - \frac{4}{\sqrt{3}} - \frac{2}{\sqrt{3}} = -\frac{10}{\sqrt{3}} \approx -5.77 \frac{\text{g/gal}}{\text{m}}$$

Ex: Find the equation of the tangent plane to the ellipsoid $x^2 + 2y^2 + z^2 = 15$ at $(2, 1, 3)$.

The ellipsoid is the level surface of

$$F(x, y, z) = x^2 + 2y^2 + z^2$$



$$\vec{n} = \nabla F(2, 1, 3)$$

$$\nabla F = 2x\vec{i} + 4y\vec{j} + 2z\vec{k}$$

$$\vec{n} = \nabla F(2, 1, 3) = 4\vec{i} + 4\vec{j} + 6\vec{k}$$

$$P = (2, 1, 3)$$

Plane: $4(x-2) + 4(y-1) + 6(z-3) = 0.$
