Let $z = f(x, y)$. We defined partial derivatives functions denoted:

$$f_x(x, y) = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} [f(x, y)] = z_x = \frac{\partial f}{\partial x} = f_x$$

$$f_y(x, y) = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y} [f(x, y)] = z_y = \frac{\partial f}{\partial y} = f_y$$

Computing partials algebraically is easy; we consider one variable to be a constant and differentiate with respect to the other.

**Ex:** Let $f(x, y) = x^3 + 3x^2y + y^2$. Find $f_x(x, y)$ and $f_y(x, y)$.

$$f_x(x, y) = \frac{\partial}{\partial x} [x^3 + 3x^2y + y^2] = 3x^2 + 6xy$$

\(\uparrow \text{y constant, take the derivative in } x.\)

$$f_y(x, y) = \frac{\partial}{\partial y} [x^3 + 3x^2y + y^2] = 3x^2 + 2y$$

\(\uparrow x \text{ constant, take the derivative in } y.\)

Geometrically, we found slopes of $z = f(x, y)$ in the $x$ and $y$ directions at any point $(x, y)$:

\[ z = f(x, y) \]
\[ f(t, s) = t^2 e^{t+s} \]
\[ \frac{\partial f}{\partial t} = \frac{\partial}{\partial t} \left[ t^2 e^{t+s} \right] = 2te^{t+s} + t^2 e^{t+s} \]
\[ \frac{\partial f}{\partial s} = \frac{\partial}{\partial s} \left[ t^2 e^{t+s} \right] = t^2 te^{t+s} = t^3 e^{t+s} \]

**Practice!**

Since partial derivatives are ordinary derivatives of cross-sections, they can be interpreted as rates of change.

Cross-section \( f(a, y) \)

Cross-section \( f(x, b) \)

\[ z = f(x, y) \]

\[ f_x(a, b) = \frac{d}{dx} \left. \left[ f(x, b) \right] \right|_{x=a} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x, b) - f(a, b)}{\Delta x} \]

\[ f_x(a, b) \quad \text{units of } z \quad \text{rate of change of } z \text{ with respect to } x \]

At \( x = a \) with \( y \) fixed at \( y = b \).
Since
\[ f_x(a,b) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x, b) - f(a,b)}{\Delta x} \]

we can approximate:

\[ f_x(a,b) \approx \frac{f(a+\Delta x, b) - f(a,b)}{\Delta x} \text{ for } \Delta x \text{ "small".} \]

Often used if we have a contour diagram or a table of values for \( f(x, y) \).

All goes the same for \( f_y(a,b) \):

\[ f_y(a,b) = \frac{d}{dy} \left[ f(a,y) \right] = \lim_{\Delta y \to 0} \frac{f(a, b+\Delta y) - f(a,b)}{\Delta y} \]

\[ f_y(a,b) \text{ units of } z \text{ units of } y \text{ - rate of change of } z \text{ with respect to } y \text{ at } y = b \text{ with } x = a \text{ fixed.} \]

\[ f_y(a,b) \approx \frac{f(a, b+\Delta y) - f(a,b)}{\Delta y} \text{ for } \Delta y \text{ "small".} \]

The approximation formulas (*) give:

\[ f(a+\Delta x, b) \approx f(a,b) + f_x(a,b) \cdot \Delta x \text{ for } \Delta x \text{ "small".} \]

\[ f(a, b_0+\Delta y) \approx f(a,b) + f_y(a,b) \cdot \Delta y \text{ for } \Delta y \text{ "small".} \]

All as in MTH 141 as one variable is fixed.
Suppose that your weight, \(w\), in pounds, is a function \(w = f(c,m)\) of the number of calories, \(c\), you consume daily and the number of minutes, \(m\), you exercise daily.

(a) What are the units of \(\frac{\partial w}{\partial c}(c,m)\)?

(b) What is the practical meaning of the statement \(\frac{\partial w}{\partial c}(2100, 20) = 0.007\)?

(c) What are the units of \(\frac{\partial w}{\partial m}(c,m)\)?

(d) What is the practical meaning of the statement \(\frac{\partial w}{\partial m}(2100, 20) = -0.2\)?

(e) Assume that \(w(2100, 20) = 120, \frac{\partial w}{\partial m}(2100, 20) = -0.2\). Estimate \(w(2100, 25)\).

\[\begin{align*}
\frac{\partial w}{\partial c} (c, m) &\quad \text{lb} \quad \text{cal} \\
\frac{\partial w}{\partial m} (c, m) &\quad \text{min} \\
\end{align*}\]

(a) \(\frac{\partial w}{\partial c} (c, m) = \frac{\text{lb}}{\text{cal}}\) (with minutes of exercise fixed, how fast is your weight increasing when number of calories increases.)

(b) \(\frac{\partial w}{\partial c} (2100, 20) = 0.007 \frac{\text{lb}}{\text{cal}}\)

When you exercise 20 min daily and consume 2100 calories daily, your weight is increasing at the (instantaneous) rate of 0.007 \(\frac{\text{lb}}{\text{cal}}\) if you increase your calories (and keep 20 min constant).

(c) \(\frac{\partial w}{\partial m} (c, m) = \frac{\text{lb}}{\text{min}}\) (with calories fixed how fast is your weight decreasing as you increase minutes of exercise.)
(d) \( \frac{\partial W}{\partial m} (2100, 20) = -0.2 \frac{\text{lb}}{\text{min}} \)

When you exercise 20 min daily and consume 2100 calories your weight is changing at the (instantaneous) rate of \(-0.2 \frac{\text{lb}}{\text{min}}\) as you increase minutes of exercise. In other words, for each additional minute you will loose approximately 0.2 lb.

(c) \( W(2100, 20) = 120 \) cal, \( \frac{\partial W}{\partial m}(2100, 20) = -0.2 \frac{\text{lb}}{\text{min}} \)

Given that, estimate:

\[ W(2100, 25) \approx \]

\[ W(2100, 25) \approx 120 \text{lb} + (-0.2) \frac{\text{lb}}{\text{min}} \cdot 5 \text{min} = 119 \text{lb} \]

Approximately as \( \frac{\partial W}{\partial m}(2100, 20) = -0.2 \frac{\text{lb}}{\text{min}} \) is the instantaneous rate at \( c = 2100 \) and \( m = 20 \) and it may not stay constant between \( (2100, 20) \) and \( (2100, 25) \).
For Problems (a) and (b) refer to Table 9.5 giving the wind-chill factor, \( C \) in °F, as a function \( f(w, T) \) of the wind speed, \( w \), and the temperature, \( T \). The wind-chill factor is a temperature which tells you how cold it feels, as a result of the combination of wind and temperature.

**TABLE 9.5 Wind-chill factor (°F)**

<table>
<thead>
<tr>
<th>mph</th>
<th>35</th>
<th>30</th>
<th>25</th>
<th>20</th>
<th>15</th>
<th>10</th>
<th>5</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>33</td>
<td>27</td>
<td>21</td>
<td>16</td>
<td>12</td>
<td>7</td>
<td>0</td>
<td>-5</td>
</tr>
<tr>
<td>10</td>
<td>22</td>
<td>16</td>
<td>(10)</td>
<td>3</td>
<td>-3</td>
<td>-9</td>
<td>-15</td>
<td>-22</td>
</tr>
<tr>
<td>15</td>
<td>16</td>
<td>9</td>
<td>2</td>
<td>-5</td>
<td>-11</td>
<td>-18</td>
<td>-25</td>
<td>-31</td>
</tr>
<tr>
<td>20</td>
<td>12</td>
<td>4</td>
<td>-3</td>
<td>-10</td>
<td>-17</td>
<td>-24</td>
<td>-31</td>
<td>-39</td>
</tr>
<tr>
<td>25</td>
<td>8</td>
<td>1</td>
<td>-7</td>
<td>-15</td>
<td>-22</td>
<td>-29</td>
<td>-36</td>
<td>-44</td>
</tr>
</tbody>
</table>

(a) Estimate \( f_w(10, 25) \). What does your answer mean in practical terms?

(b) Estimate \( f_T(5, 20) \). What does your answer mean in practical terms?

\[
C = f(w, T)
\]

\( T \) stays constant at 25°F, wind speed changes.

The best estimate:

Increasing \( w \):

\[
\frac{2^\circ F - 10^\circ F}{5 \text{ mph}} = -\frac{8}{5} \frac{\circ F}{\text{mph}}
\]

Decreasing \( w \):

\[
\frac{10^\circ F - 21^\circ F}{5 \text{ mph}} = -\frac{11}{5} \frac{\circ F}{\text{mph}}
\]

\[
f_w(10, 25) \approx -\frac{8}{5} \frac{\circ F}{\text{mph}} = -1.6 \frac{\circ F}{\text{mph}}
\]

\[
\approx -\frac{11}{5} \frac{\circ F}{\text{mph}} = -2.2 \frac{\circ F}{\text{mph}}
\]

Or average them:

\[
f_w(10, 25) \approx -\frac{3.8}{2} = 1.9 \frac{\circ F}{\text{mph}}
\]
Similarly partial derivatives can be estimated for a given function \( z = f(x, y) \) from its contour diagram. Do practice problems for 14.1, 14.2.

14.3 **Differentiability, Tangent Plane, Local Linearization**

A function of one variable, \( y = f(x) \), is called differentiable at \( x = a \) if \( f'(a) \) exists which is when the graph \( y = f(x) \) has the tangent line at \((a, f(a))\):

![Tangent Line Diagram]

\[ f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \]

The tangent line exists if the graph of \( y = f(x) \) "flattens" to a straight line when we look at smaller and smaller portions of the graph around the point \( P = (a, f(a)) \):

![Tangent Line Diagram]
Not every curve around each point "flattens" to a straight line:

\[ f(x) = |x| \]
\[ \text{at } a = 0: \]

No tangent line.

Curves that "flatten" to straight lines as we look at smaller and smaller portions are called "smooth".

How about surfaces in 3D and graphs of functions \( z = f(x, y) \)? A "nice", "smooth" surface "flattens" to a plane when we look at small portions of it:

That plane it calls the tangent plane.
Let \( z = f(x, y) \), \((x, y) = (a, b)\) be given. \( f(x, y) \) is called differentiable at \((a, b)\) if 
\( z = f(x, y) \) has a tangent plane at \((a, b, f(a, b))\).

Let's denote by \( L(x, y) \) the linear function whose graph 
\( z = L(x, y) \) is the tangent plane.

What is the formula for \( L(x, y) \); that is, 
what is the equation of the tangent plane?

The tangent lines to the cross-sections \( f(x, b) \) and \( f(a, y) \) have to be on the tangent plane. Those tangent lines have slopes \( f_x(a, b) \) and \( f_y(a, b) \). Thus, 
the slope of \( z = L(x, y) \) in the \( x \)-direction is \( f_x(a, b) \), the slope in the \( y \)-direction is \( f_y(a, b) \). The plane passes through \((a, b, f(a, b))\). Thus, the equation of the tangent plane at \((a, b)\) is:
The linear function whose graph is the tangent plane:
\[ L(x, y) = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b). \]

Clearly:
\[ f(x, y) \approx L(x, y) \text{ for } (x, y) \text{ close to } (a, b). \]

\( L(x, y) \) is called the local linearization of \( f(x, y) \) at \( (a, b) \).

\[ \text{Ex: Find the equation of the tangent plane (assuming it exists) to} \]
\[ z = f(x, y), \quad f(x, y) = ye^{\frac{x}{y}} \text{ at } (1, 1, e). \]

\[ z = e + f_x(1, 1)(x-1) + f_y(1, 1)(y-1) \]
\[ f_x = \frac{\partial}{\partial x} \left[ ye^{\frac{x}{y}} \right] = e^{\frac{x}{y}}, \quad f_x(1, 1) = e \]
\[ f_y = \frac{\partial}{\partial y} \left[ ye^{\frac{x}{y}} \right] = e^{\frac{x}{y}} + y \cdot \left( \frac{f_x}{y^2} \right) e^{\frac{x}{y}} = e^{\frac{x}{y}} - \frac{x}{y} e^{\frac{x}{y}} \]
\[ f_y(1, 1) = e - e = 0 \]

\[ L: \quad z = e + e(x-1) = e \times e = e \]

\[ z = ex \]
Since \( f(x, y) \approx L(x, y) \) for \((x, y)\) close to \((a, b)\), we have the following approximation formula:

\[
\begin{aligned}
\quad f(x, y) & \approx f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) \\
\text{for (x-a), (y-b) "small".}
\end{aligned}
\]

In other words:

\[
f(a+\Delta x, b+\Delta y) \approx f(a, b) + f_x(a, b)\Delta x + f_y(a, b)\Delta y,
\]

\[
\Delta x = x-a, \quad \Delta y = y-b.
\]

**Ex:** An unevenly heated plate has temperature \( T(x, y) \) in °C at the point \((x, y)\). If \( T(2,1) = 135\), \( T_x(2,1) = 16\), \( T_y(2,1) = -15\), estimate the temperature at \((2.04, 0.97)\).

\[
\begin{aligned}
T(2.04, 0.97) & \approx T(2,1) + T_x(2,1)\cdot\Delta x + T_y(2,1)\cdot\Delta y \\
\Delta x & = 0.04, \quad \Delta y = -0.03 \\
& \approx 135 + 16 \cdot 0.04 + (-15) \cdot (-0.03) = 136.09 \text{ °C}
\end{aligned}
\]
The Differential

Let \( z = f(x, y) \). We have:

\[
f(x, y) = f(a, b) + f_x(a, b) \cdot \Delta x + f_y(a, b) \cdot \Delta y
\]

Denote

\[
\Delta f = f(x, y) - f(a, b).
\]

Then:

\[
\Delta z = \Delta f = f_x(a, b) \cdot \Delta x + f_y(a, b) \cdot \Delta y
\]

The infinitesimal version of this formula is called the differential, \( df \) or \( dz \), of \( f(x, y) \) at \((a, b)\):

\[
df = f_x(a, b) \, dx + f_y(a, b) \, dy
\]

In general:

\[
df = f_x \, dx + f_y \, dy
\]

---

**Ex.** Compute the differential of

\[
z = x \sin(xy).
\]

\[
\frac{\partial z}{\partial x} = \sin(xy) + xy \cos(xy), \quad \frac{\partial z}{\partial y} = x^2 \cos(xy)
\]

Thus:

\[
\,dz = (\sin(xy) + xy \cos(xy)) \, dx + (x^2 \cos(xy)) \, dy
\]
When is a given function \( z = f(x, y) \) differentiable at \((a, b)\)? Is it enough that \( f_x(a, b), f_y(a, b) \) exist? Not really. Let's look at a couple of examples.

\[ \begin{align*}
\mathbf{Ex} & \quad z = f(x, y), \quad f(x, y) = 1y. \\
& \quad \begin{array}{ll}
f_x(0,0) = ? & f_y(0,0) = ? \\
\end{array}
\end{align*} \]

Look at cross-sections:
\( f(x,0), \quad f(0,y). \)

Clearly, \( f_x(0,0) = 0, \quad f_y(0,0) \) does not exist. No tangent plane at \((0,0)\) (or any point \((a, 0)\)).

It can happen, however, that \( f_x(a, b), \quad f_y(a, b) \) both exist and there is no tangent plane:

\[ \begin{align*}
& \quad z = f(x, y). \quad \text{What about } f_x(0,0), \quad f_y(0,0) ? \\
& \quad \text{What about differentiability of } \quad f(x, y) \text{ at } (0,0)? \\
\end{align*} \]

Differentiability for \( f(x, y) \) is a bit complicated.