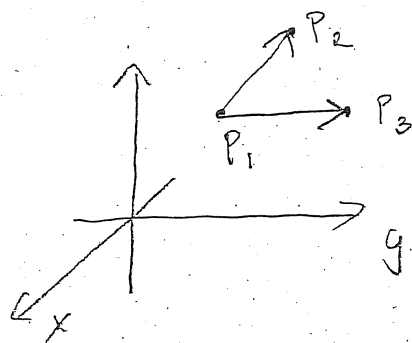


Ex: Find an equation of the plane passing through points

$$P_1 = (1, 1, 1), \quad P_2 = (0, 1, 2), \quad P_3 = (2, -5, 0)$$

Find a unit normal vector to the plane.



We have a point, we need a normal vector.

A normal will have to be $\perp$ to $\vec{P_1P_2}$ and $\vec{P_1P_3}$.

$$\vec{n} = (\vec{P_1P_2}) \times (\vec{P_1P_3})$$

$$\vec{P_1P_2} = (-1, 0, 1), \quad \vec{P_1P_3} = (1, -6, -1)$$

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 1 \\ 1 & -6 & -1 \end{vmatrix} = \vec{i} \cdot 6 - \vec{j} \cdot 0 + \vec{k} \cdot 6$$

$$\vec{n} = 6\vec{i} + 6\vec{k}$$

(c) Equation (using P_1):

$$6(x-1) + 6(z-1) = 0$$

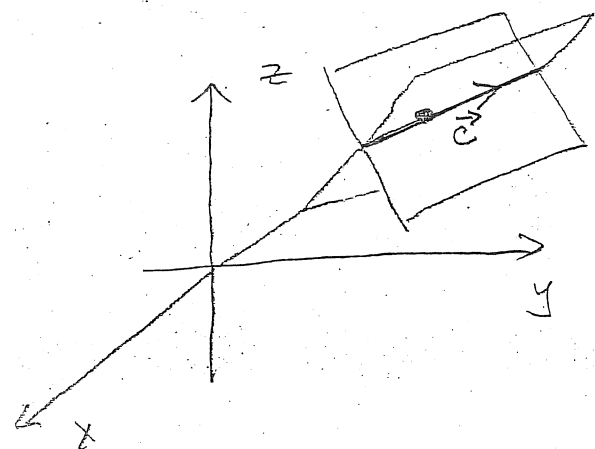
$$(b) \quad \vec{u}_n = \frac{\vec{n}}{\|\vec{n}\|} = \frac{6}{6\sqrt{2}}\vec{i} + \frac{6}{6\sqrt{2}}\vec{k} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{k}$$

$$\|\vec{n}\| = \sqrt{36+36} = 6\sqrt{2}$$

Ex : Find a vector parallel to the intersection of the planes

$$L_1: 2x - 3y + 5z = 2$$

$$L_2: 4x + y - 3z = 7$$



$$\vec{c} \perp n_{L_1}, \quad \vec{c} \perp n_{L_2}$$

$$n_{L_1} = (2, -3, 5)$$

$$n_{L_2} = (4, 1, -3)$$

$$\vec{c} = n_{L_1} \times n_{L_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 5 \\ 4 & 1 & -3 \end{vmatrix} =$$

$$= 4\vec{i} - (-26)\vec{j} + 14\vec{k} = \underline{4\vec{i} + 26\vec{j} + 14\vec{k}}$$

14.1, 14.2 Partial Derivatives

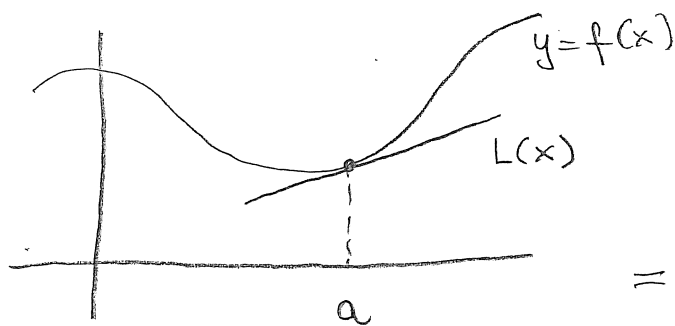
Going back to functions of two variables:

$$z = f(x, y).$$

Today we are going to define and interpret partial derivatives of $f(x, y)$ with respect to x and y .

Recall for a function of one variable:

$$y = f(x)$$



$$f'(a) = \left. \frac{dy}{dx} \right|_{x=a} = m_{\text{tan}}$$

$$= \left(\begin{array}{c} \text{The rate of change} \\ \text{of } y \text{ at } x=a \end{array} \right) \frac{\text{units of } y}{\text{unit of } x}$$

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

So
$$f(x) \approx f(a) + f'(a)(x-a) \quad \leftarrow (x = a+h)$$

The equation of the tangent line:

$$L(x) = f(a) + f'(a)(x-a)$$

$L(x)$ local linearization of $f(x)$ at $x=a$.

$f(x)$ differentiable at $x=a \Leftrightarrow$ the tangent line exists.

How much of that translates to $f(x, y)$ and how?

The first step : partial derivatives.

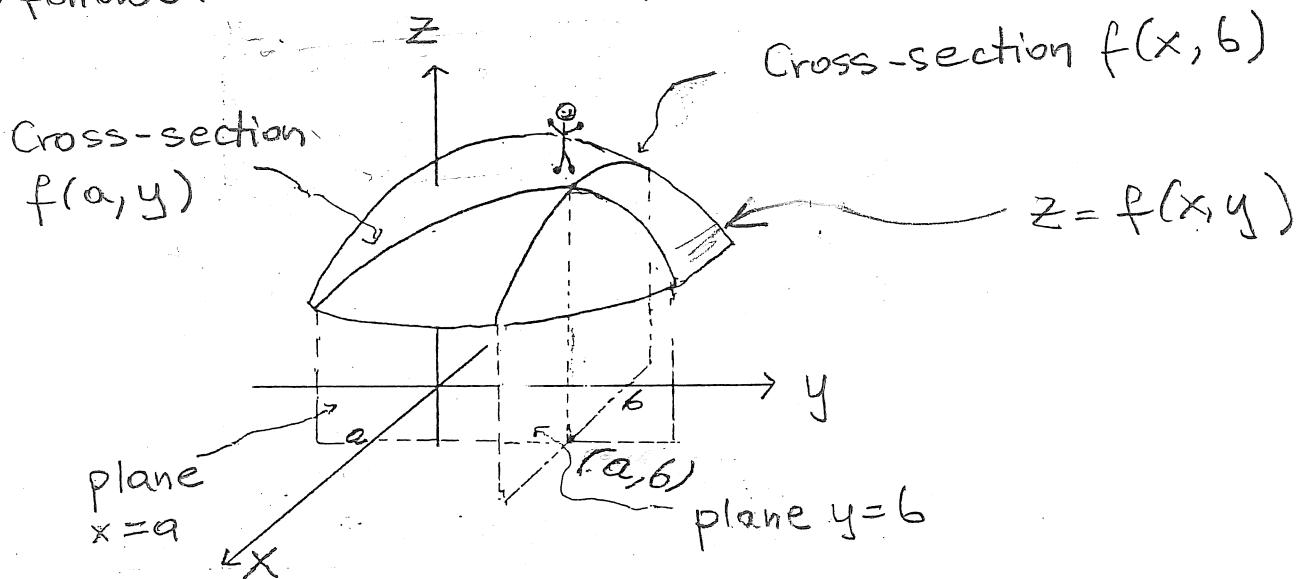
Let a function $z = f(x, y)$ and a point (a, b) in its domain be given. We define :

$f_x(a, b)$ - the partial derivative of $f(x, y)$ with respect to x at (a, b)

and

$f_y(a, b)$ - the partial derivative of $f(x, y)$ with respect to y at (a, b) .

as follows.



$$f_x(a, b) = \frac{d}{dx} \Big|_{x=a} [f(x, b)]$$

$$f_y(a, b) = \frac{d}{dy} \Big|_{y=b} [f(a, y)]$$

In other words

$f_x(a, b)$ = the slope of $f(x, b)$ at $x=a$.

$f_y(a, b)$ = the slope of $f(a, y)$ at $y=b$.

So to calculate

$$f_x(a, b)$$

we fix $y = b$, consider the function $f(x, b)$ which is a function of one variable x , and calculate the ordinary derivative of $f(x, b)$ with respect to x .

Similarly, to calculate

$$f_y(a, b)$$

we fix $x = a$, consider $f(a, y)$ etc.

Ex: Let $f(x, y) = x^2 + y^3$. Find

$$f_x(2, 1), f_x(1, 3).$$

To find $f_x(2, 1)$, we fix $y = 1$ and consider the cross-section

$$f(x, 1) = x^2 + 1$$

Now we take the ordinary derivative

$$\frac{d}{dx} [f(x, 1)] = \frac{d}{dx} [x^2 + 1] = 2x.$$

And evaluate at $x = 2$: $f_x(2, 1) = 2x|_{x=2} = 4$

For $f_x(1, 3)$, we take

$$f(x, 3) = x^2 + 27$$

$$\frac{d}{dx} \Big|_{x=1} [x^2 + 27] = 2x|_{x=1} = 2.$$

Clearly calculating $f_x(a, b)$, $f_y(a, b)$ for each given point separately is silly. For a given

$$f(x, y)$$

we should calculate partial derivatives functions:

$$f_x(x, y), f_y(x, y)$$

and then evaluate at any point (a, b) we want. How to do this?

Ex: Let $f(x, y) = x^2 + y^3$. Find

$f_x(x, y)$ and $f_y(x, y)$. Find $f_y(2, 1)$.

To find $f_x(x, y)$, we take $f(x, y) = x^2 + y^3$ and assume that y is fixed so y is a constant.

Assuming that y is a constant, we take the derivative of $f(x, y) = x^2 + y^3$ with respect to x :

$$f_x(x, y) = (x^2 + y^3)_x = 2x$$

To find $f_y(x, y)$, we assume that x is a constant and differentiate with respect to y :

$$f_y(x, y) = (x^2 + y^3)_y = 3y^2.$$

So $f_y(2, 1) = 3$. Easy!

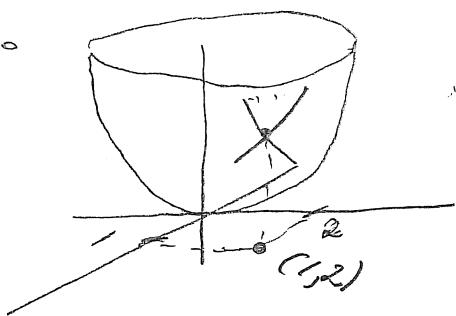
Ex : Let $f(x, y) = x^2 + y^2$. Find

$$f_x(1, 2), \quad f_y(1, 2).$$

$$f_x(x, y) = 2x, \quad f_y(x, y) = 2y$$

$$f_x(1, 2) = 2, \quad f_y(1, 2) = 4.$$

So



For the paraboloid at $(1, 2)$
the slope in the x direction
is 2, the slope in the y direction
is 4.

Leibnitz Notation:

$$z = f(x, y), \quad f_x(x, y) = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} [f(x, y)],$$

$$f_y(x, y) = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y} [f(x, y)]$$

$$f_x(a, b) = \frac{\partial z}{\partial x} \bigg|_{(a, b)}, \quad f_y(a, b) = \frac{\partial z}{\partial y} \bigg|_{(a, b)}$$

Ex : $z = x^2 y$. Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} [x^2 y] = 2yx, \quad \frac{\partial z}{\partial y} = \frac{\partial}{\partial y} [x^2 y] = x^2.$$

For a given $z = f(x, y)$, find partial derivatives $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

Ex: $z = \sin(xy^3)$.

$$\frac{\partial z}{\partial x} = \cos(xy^3) \cdot y^3 \quad (\sin(x \cdot 5))'_x$$

$$\frac{\partial z}{\partial y} = \cos(xy^3) \cdot 3xy^2 \quad (\sin(5y^3))'_x$$

Ex: $z = x^2 e^{xy}$

$$\frac{\partial z}{\partial x} = 2x e^{xy} + x^2 y e^{xy}$$

$$\frac{\partial z}{\partial y} = x^2 \cdot x e^{xy} = x^3 e^{xy}$$

Practice!

Of course partial derivatives, $f_x(a, b)$, $f_y(a, b)$ are rates of change of $f(x, y)$ at (a, b) in the x direction and the y direction.