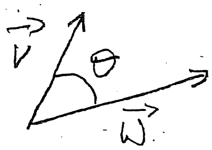
13.3 The Dot Product, cont'd

Recall for any:

$$\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}, \quad \vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$$

the dot product $\vec{v}\cdot\vec{w}$ can be defined algebraically or geometrically:

① $\vec{v}\cdot\vec{w} = v_1\cdot w_1 + v_2 w_2 + v_3\cdot w_3$



② $\vec{v}\cdot\vec{w} = \|\vec{v}\|\cdot\|\vec{w}\|\cos\theta$,

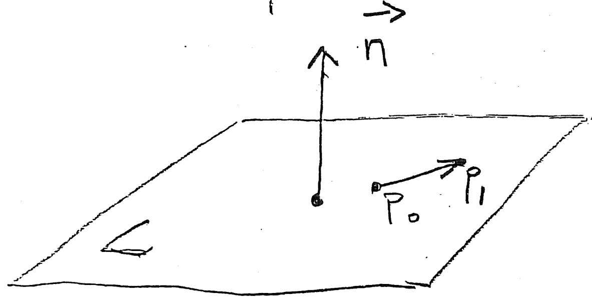
where $0 \le \theta \le \pi$ is the angle between $\vec{v}$ and $\vec{w}$.

$\vec{v}$ is perpendicular to $\vec{w}$, $\vec{v} \perp \vec{w}$, if and only if

$$\vec{v}\cdot\vec{w} = 0$$

A Normal Vector to a Plane

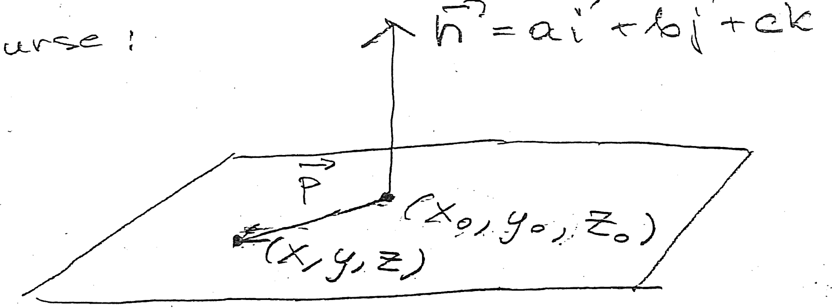
Def: $\vec{n}$ is a normal vector to a plane L if $\vec{n}$ is perpendicular to any vector $\overrightarrow{P_0P_1}$ for any two points P_0, P_1 on the plane.



Remark: If $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$ is normal to a plane L, $P_0 = (x_0, y_0, z_0)$ is a point on L, then the equation of L is:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Of course:



$$\vec{P} = (x-x_0)\vec{i} + (y-y_0)\vec{j} + (z-z_0)\vec{k}$$

$$\vec{P} \cdot \vec{n} = a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

We can rewrite:

$$ax + by + cz = d \quad , \quad \vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$$

Ex 1: Find the equation of the plane, L_0, through $P_0 = (-2, 3, 2)$ and parallel to the plane

L_1: $3x + y + z = 4$.

Sol. Two planes are parallel if they normal vectors are the same. Normal to L_1 is:

$$\vec{n} = 3i + \vec{j} + \vec{k} \quad , \quad P_0 = (-2, 3, 2).$$

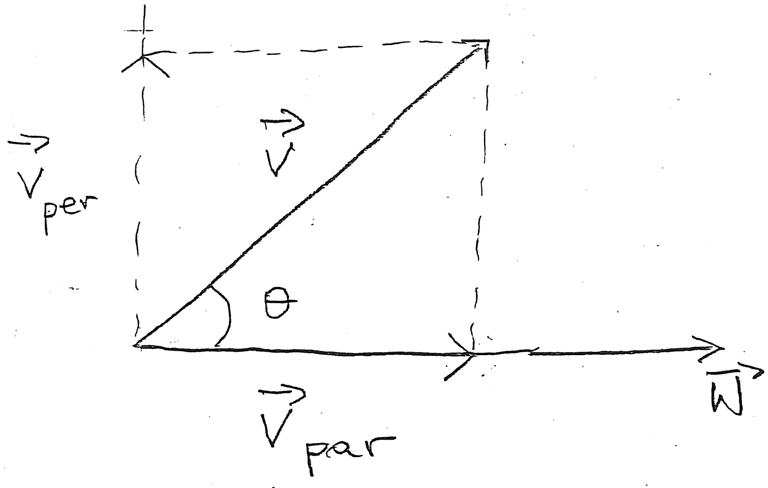
So:

$$L_0: \quad 3(x+2) + (y-3) + (z-2) = 0$$

The dot product can be used to find parallel and perpendicular components of one vector with respect to another vector.

Very important in physics.

Let $\vec{v}$, $\vec{w}$ be two vectors in 3d.



Let $\vec{u} = \dfrac{\vec{w}}{\|\vec{w}\|}$. (The unit vector in the direction of $\vec{w}$.)

We have:

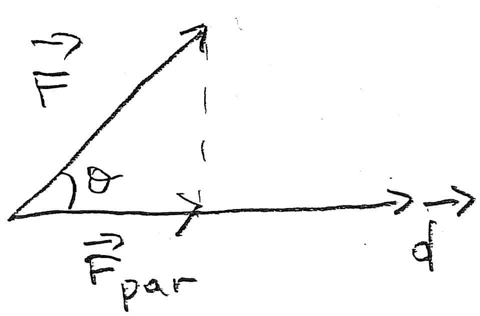
$$\|\vec{v}_{par}\| = \|\vec{v}\| \cdot \cos\theta = \|\vec{u}\| \, \|\vec{v}\| \cos\theta = \vec{v} \cdot \vec{u}$$

Thus:

$$\vec{v}_{par} = (\vec{v} \cdot \vec{u})\,\vec{u}, \quad \vec{v}_{per} = \vec{v} - \vec{v}_{par}.$$

$\vec{v}_{par}$ - projection of $\vec{v}$ onto $\vec{w}$.

Work: $W = F \cdot d$ if force and displacement are in the same direction.



$$W = \|\vec{F}_{par}\| \cdot \|\vec{d}\|$$

$$\|\vec{F}_{par}\| = \|\vec{F}\| \cdot \cos\theta = \frac{\vec{F} \cdot \vec{d}}{\|\vec{d}\|}$$

So $\underline{W = \vec{F} \cdot \vec{d}}$

Ex: Write $\vec{a} = 3\vec{i} + 2\vec{j} - 6\vec{k}$ as the sum of two vectors, one parallel, and one perpendicular to $\vec{d} = 2\vec{i} - 4\vec{j} + \vec{k}$.

Sol:

$$\vec{a} = \vec{a}_{par} + \vec{a}_{per}$$

$$\vec{a}_{par} = \left(\vec{a} \cdot \frac{\vec{d}}{\|\vec{d}\|}\right) \cdot \frac{\vec{d}}{\|\vec{d}\|} \quad , \quad \vec{u} = \frac{\vec{d}}{\|\vec{d}\|}$$

$$\|\vec{d}\| = \sqrt{4+16+1} = \sqrt{21}$$

$$\vec{u} = \frac{2}{\sqrt{21}}\vec{i} - \frac{4}{\sqrt{21}}\vec{j} + \frac{1}{\sqrt{21}}\vec{k}.$$

$$\vec{a}_{par} = \underbrace{\left(\frac{6}{\sqrt{21}} - \frac{8}{\sqrt{21}} - \frac{6}{\sqrt{21}}\right)}_{\vec{a}\cdot\vec{u}} \cdot \left(\frac{2}{\sqrt{21}}\vec{i} - \frac{4}{\sqrt{21}}\vec{j} + \frac{1}{\sqrt{21}}\vec{k}\right) =$$

$$= -\frac{16}{21}\vec{i} + \frac{32}{21}\vec{j} - \frac{8}{21}\vec{k} = \vec{a}_{par} = \frac{8}{21}\vec{d}$$

$$\vec{a}_{per} = \vec{a} - \vec{a}_{par} = (3\vec{i} + 2\vec{j} - 6\vec{k}) - \left(-\frac{16}{21}\vec{i} + \frac{32}{21}\vec{j} - \frac{8}{21}\vec{k}\right) =$$

$$= \frac{79}{21}\vec{i} + \frac{10}{21}\vec{j} - \frac{118}{21}\vec{k} = \vec{a}_{per}$$

$$\vec{a} = \vec{a}_{par} + \vec{a}_{per}$$

13.4 Cross Product

Let $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$, $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$

be given vectors.

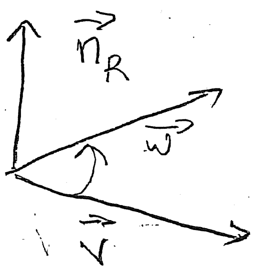
$\vec{v} \cdot \vec{w}$ is a number.

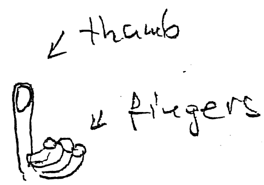
$\vec{v} \times \vec{w}$ is a vector.

$\vec{v} \times \vec{w}$ can be defined geometrically or algebraically.

A few remarks first.

The right-hand unit normal to $\vec{v}$ and $\vec{w}$





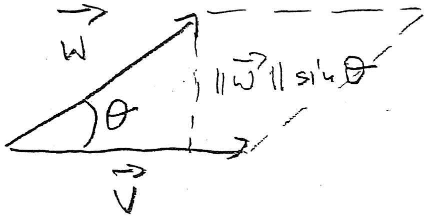
Curl the fingers of your right hand through the smaller of the two angles between $\vec{v}$ and $\vec{w}$ in the direction of $\vec{w}$. Your thumb is pointing toward $\vec{n}_R$.

$\vec{n}_R \perp \vec{v}$, $\vec{n}_R \perp \vec{w}$

$\|\vec{n}_R\| = 1$.

$\vec{v} \times \vec{w}$ points toward $\vec{n}_R$.

The area of a parallelogram

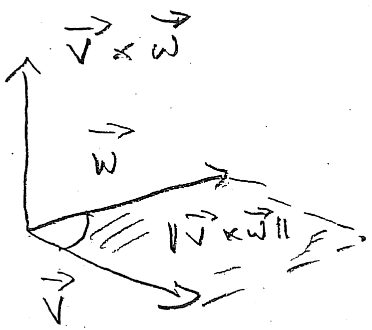


$A = \|\vec{w}\| \cdot \|\vec{v}\| \cdot \sin\theta$

① $\vec{v} \times \vec{w}$ geometrically

$$\vec{v} \times \vec{w} = (\|\vec{v}\| \, \|\vec{w}\| \sin\theta)\, \vec{n}_R \,,$$

where $0 \le \theta \le \pi$ is the angle between $\vec{v}$ and $\vec{w}$,
$\vec{n}_R$ the right-hand unit normal vector to $\vec{v}$ and $\vec{v}$.



$$(\vec{v} \times \vec{w}) \perp \vec{v}$$
$$(\vec{v} \times \vec{w}) \perp \vec{w}$$
$$\vec{v},\ \vec{w},\ \vec{v} \times \vec{w} \text{ right-handed}$$
$$\|\vec{v} \times \vec{w}\| = A.$$

② $\vec{v} \times \vec{w}$ algebraically

$$\vec{v} \times \vec{w} = (v_2 w_3 - v_3 w_2)\vec{i} + (v_3 w_1 - v_1 w_3)\vec{j} + (v_1 w_2 - v_2 w_1)\vec{k}.$$

Impossible to remember. Easy in terms of determinants.

Recall: 2×2 determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

3 × 3 determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \overset{*}{-} a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

To compute $\vec{v} \times \vec{w}$:

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \vec{i} \cdot (v_2 w_3 - v_3 w_2) - \vec{j}\,(v_1 w_3 - v_3 w_1) + \vec{k} \cdot (v_1 w_2 - v_2 w_1) =$$

$$= \vec{i} \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \vec{j} \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + \vec{k} \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}.$$

Remark: If $\vec{v} \parallel \vec{w}$, $\vec{v} \times \vec{w} = \vec{0}$

Ex: Let $\vec{v} = 2\vec{i} + \vec{j} - 2\vec{k}$, $\vec{w} = 3\vec{i} + \vec{k}$.

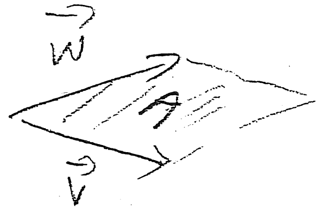
(a) Find $\vec{v} \times \vec{w}$

(b) Find the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$.

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ 3 & 0 & 1 \end{vmatrix} = \vec{i} \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} =$$

$$= \vec{i} - 8\vec{j} - 3\vec{k}$$

(b)



$$A = \|\vec{v} \times \vec{w}\| = \sqrt{1 + 64 + 9} \approx 8.6$$