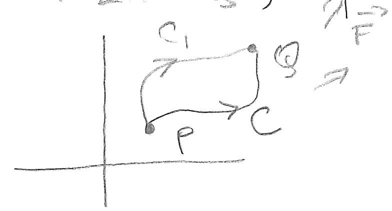
1) No quiz R
2) Final

Apr 21, 23

18.3 Path-independent Vector Fields

<u>Def</u>: A vector field $\vec{F}$ is said to be path-independent (or conservative), if for any two points P and Q, the line integral



$$\int_C \vec{F}\,d\vec{r}$$

is the same for any path C from P to Q. ▲

If $\vec{F}$ is conservative, we can simply write:

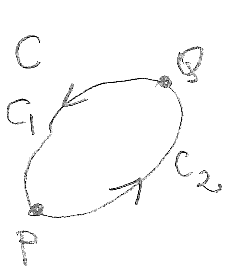
$$\int_P^Q \vec{F}\,d\vec{r}\,.$$

It seems like a strange property. Are there any p-i fields? We shall soon see. The first important observation:

<u>Th</u>: $\vec{F}$ is conservative if and only if for any closed path C:

$$\oint_C \vec{F}\,d\vec{r} = 0.$$ ▲

Indeed. "⇒" Let $\vec{F}$ be conservative and C be closed, choose and two points P and Q on C, take C_1 and C_2 as on the picture. Then:

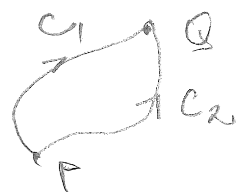


$$\int_P^Q \vec{F}\,d\vec{r} = \int_{C_2} \vec{F}\,d\vec{r} = \int_{-C_1} \vec{F}\,d\vec{r} = -\int_{C_1} \vec{F}\,d\vec{r},$$

So

$$\oint_C \vec{F}\,d\vec{r} = \int_{C_2} \vec{F}\,d\vec{r} - \left(-\int_{C_1} \vec{F}\,d\vec{r}\right) = 0.$$

Similarly "$\Leftarrow$". Suppose $\oint_C \vec{F} d\vec{r} = 0$ for any closed path C. To show that $\vec{F}$ is conservative take any P and Q and any two paths C_1, C_2 from P to Q:



Then:

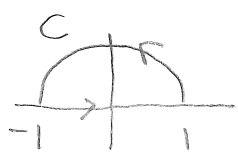
$$\int_{C_1} \vec{F} + \int_{-C_2} \vec{F} = 0 = \int_{C_1} \vec{F} - \int_{C_2} \vec{F} = 0$$

Thus: $\int_{C_1} \vec{F} d\vec{r} = \int_{C_2} \vec{F} d\vec{r}$.

So

$$\vec{F} \text{ conservative} \iff \oint_C \vec{F} d\vec{r} = 0 \text{ for any closed path.}$$

Ex: Recall: $\vec{F}(x,y) = -y\vec{i} + x\vec{j}$



$$\oint_C \vec{F} d\vec{r} = \pi.$$

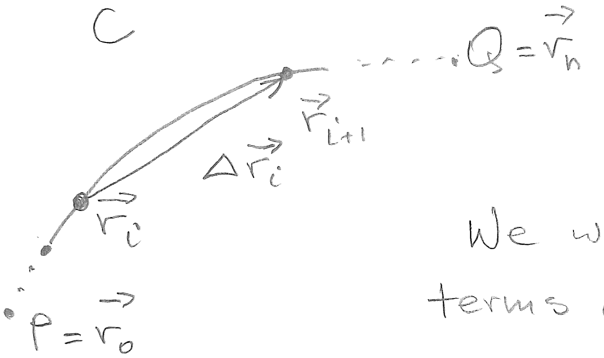
Thus $\vec{F}$ is not conservative.

So what vector fields are conservative? Gradient fields and only gradient fields. If

$$\vec{F}(x,y) = \nabla f(x,y)$$

for some $f(x,y)$, then $\vec{F}$ is conservative and vice-versa. Why? Let me give some justification.

Let $\vec{F}(x,y) = \nabla f(x,y)$ for some $f(x,y)$. Let C be an oriented smooth path from P to Q.



$$\int_C \vec{F}\,d\vec{r} = \lim_{\|\Delta\vec{r}\| \to 0} \sum_{i=0}^{n-1} \vec{F}(\vec{r}_i)\cdot\Delta\vec{r}_i$$

We will express $\vec{F}(\vec{r}_i)\cdot\Delta\vec{r}_i$ in terms of f. Observe:

$$f(\vec{r}_{i+1}) - f(\vec{r}_i) \cong f_{\frac{\Delta\vec{r}_i}{\|\Delta\vec{r}_i\|}}(\vec{r}_i)\cdot\|\Delta\vec{r}_i\|$$

The directional derivative at $\vec{r}_i$ in the direction $\frac{\Delta\vec{r}_i}{\|\Delta\vec{r}_i\|}$, i.e, the rate of change

magnitude of the displacement in the direction $\frac{\Delta\vec{r}_i}{\|\Delta\vec{r}_i\|}$

But $f_{\frac{\Delta\vec{r}_i}{\|\Delta\vec{r}_i\|}}(\vec{r}_i) = \nabla f(\vec{r}_i)\cdot\frac{\Delta\vec{r}_i}{\|\Delta\vec{r}_i\|}$. Thus

$$f(\vec{r}_{i+1}) - f(\vec{r}_i) \cong \nabla f(\vec{r}_i)\cdot\Delta\vec{r}_i = \vec{F}(\vec{r}_i)\cdot\Delta\vec{r}_i$$

Thus:

$$\sum_{i=0}^{n-1} \vec{F}(\vec{r}_i)\cdot\Delta\vec{r}_i \cong \sum_{i=0}^{n-1}\left(f(\vec{r}_{i+1}) - f(\vec{r}_i)\right) =$$

$$= f(\vec{r}_1) - f(\vec{r}_0) + f(\vec{r}_2) - f(\vec{r}_1) + f(\vec{r}_3) - f(\vec{r}_2) + \ldots + f(\vec{r}_n) - f(\vec{r}_{n-1}) =$$

$$= f(\vec{r}_n) - f(\vec{r}_0) = f(Q) - f(P)$$

Thus

$$\int_C \vec{F}\,d\vec{r} = f(Q) - f(P).$$

This is not a precise proof! Only justification.

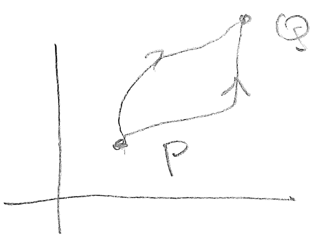
Th: A continuous vector field $\vec{F}$ defined in an open region of the xy-plane is path-independent if and only if $\vec{F}$ is a gradient field; that is:

$$\vec{F} = \nabla f$$

for some function f. In that case, for any piecewise smooth path C from P to Q we have:

$$\int_C \vec{F}\, d\vec{r} = \int_C \nabla f\, d\vec{r} = f(Q) - f(P).$$

The Fundamental Theorem for Line Integrals.



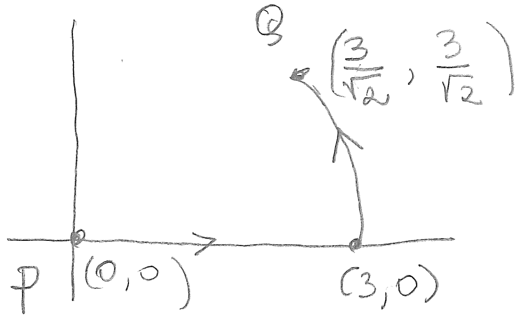
$$\int_P^Q \vec{F}\, d\vec{r} = \int_P^Q \nabla f\, d\vec{r} = f(Q) - f(P).$$

Remember the FTC from Calc I?

$$\int_a^b f'(t)\, dt = f(b) - f(a)$$

Def: If $\vec{F} = \nabla f$, then f is called <u>a potential function of $\vec{F}$</u>.

Ex: Let $\vec{F}(x,y) = x\vec{i} + y\vec{j}$, C be the path:



Find $\int_C \vec{F}\, d\vec{r}$.

We could parametrize both pieces: the segment and a piece of the circle of radius 3. But do we have to?

If $\vec{F} = \nabla f$, then $\int_C \vec{F} = f(Q) - f(P)$.

Is $\vec{F}$ a gradient field?

$$\vec{F}(x,y) = x\vec{i} + y\vec{j} \stackrel{?}{=} f_x\vec{i} + f_y\vec{j}$$

$f_x = x$, $f_y = y$. Possible.

$$f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$$

(Or $f(x,y) + C$ for any constant C.)

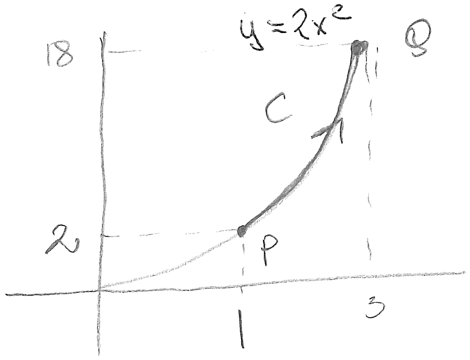
Hence:

$$\int_C \vec{F}\, d\vec{r} = \int_C \nabla f\, d\vec{r} = f(Q) - f(P) =$$

$$= \frac{1}{2}\left(\frac{3}{\sqrt{2}}\right)^2 + \frac{1}{2}\left(\frac{3}{\sqrt{2}}\right)^2 - 0 = \underline{\frac{9}{2}}.$$

Ex: $\vec{F}(x,y) = y\sin(xy)\,\vec{i} + x\sin(xy)\,\vec{j}$, C is the piece of $y = 2x^2$ from $(1,2)$ to $(3,18)$.

Find $\int_C \vec{F}\, d\vec{r}$.



We could parametrize:

$x(t) = t$

$y(t) = 2t^2$, $\quad t$ in $[1,3]$.

But maybe we can use the FTFLI ?

? $f(x,y)$? $\quad \nabla f = \vec{F}$

$f_x = y\sin(xy)$ $\quad$, $f_y = x\sin(xy)$

$$f(x,y) = -y \cdot \frac{1}{y}\cos(xy) + C(y) = -\cos(xy) + C(y)$$

$f_y = x\sin(xy) + C'(y)$. Yes. $C(y) \equiv 0$.

The potential function $\underline{f(x,y) = -\cos(xy)}$.

Thus:

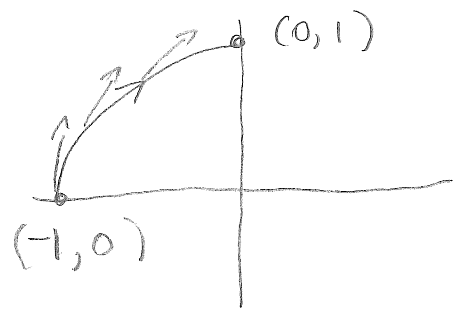
$$\int_C \vec{F}\, d\vec{r} = \int_P^Q \nabla f\, d\vec{r} = f(Q) - f(P) = -\cos(3 \cdot 18) + \cos(1 \cdot 2) =$$

$$\cong 0.4131\ldots$$

Ex: Suppose a particle subject to a force $\vec{F}(x,y) = y\vec{i} - x\vec{j}$ moves cw along the arc of the unit circle from $(-1,0)$ to $(0,1)$.

(a) Find the work done by $\vec{F}$. Explain the sign of your answer.

(b) Is $\vec{F}$ path-independent?. Explain.



$y\vec{i} - x\vec{j} \perp x\vec{i} + y\vec{j}$

Pointing cw.

(b) Is $\vec{F}$ p-i? Is it a gradient field? $\dfrac{\partial F_1}{\partial y} = 1$, $\dfrac{\partial F_2}{\partial x} = -1$. No.

$\vec{F}$ is not p-i. Also if you consider the closed path $\tilde{C}$:

$\int_{\tilde{C}} \vec{F} > 0$. So not p-i. Yet another reason.

(a) We cannot apply the FT. Hence we have to parametrize:

$\vec{r}(t) = -\cos(t)\vec{i} + \sin(t)\vec{j}$, t in $[0, \frac{\pi}{2}]$

$\vec{r}\,'(t) = \sin(t)\vec{i} + \cos(t)\vec{j}$

$\vec{F}(\vec{r}(t)) = \sin(t)\vec{i} + \cos(t)\vec{j}$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = $
$= \sin^2(t) + \cos^2(t) = 1$

$$\int_C \vec{F} = \int_0^{\frac{\pi}{2}} 1\,dt = \frac{\pi}{2}.$$

18.4 Green's Theorem, The Curl Test

We know that gradient and only gradient fields are conservative. We also know: if

$$\vec{F}(x,y) = F_1(x,y)\vec{i} + F_2(x,y)\vec{j}$$

and

$$\frac{\partial F_1}{\partial y} \neq \frac{\partial F_2}{\partial x},$$

then $\vec{F}$ is not a gradient field. Is it true: if $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ then $\vec{F}$ is a gradient field?

It would be nice. We could always decide if $\vec{F}$ is conservative by this simple test.

<u>Def</u>: Let $\vec{F}(x,y) = F_1(x,y)\vec{i} + F_2(x,y)\vec{j}$.

The (scalar) <u>curl</u> of $\vec{F}$ is defined as

$$\operatorname{curl} \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$

Note:

$$\operatorname{curl} \vec{F}(x,y) = \frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y),$$

$\operatorname{curl} \vec{F}$ is a function of (x,y).

We know already that if $\operatorname{curl} \vec{F} \neq 0$, then $\vec{F}$ is not conservative. Let's state it a bit more precisely.

Th: Let $\vec{F}(x,y) = F_1(x,y)\vec{i} + F_2(x,y)\vec{j}$ be a vector field with continuous partial derivatives of F_1 and F_2. If $\vec{F}$ is path-independent, then

$$\operatorname{curl} \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0.$$

Proof: $\vec{F}$ is path-independent if $\vec{F}$ is a gradient field. Thus for some $f(x,y)$:

$$\vec{F} = F_1\vec{i} + F_2\vec{j} = f_x\vec{i} + f_y\vec{j}.$$

Then

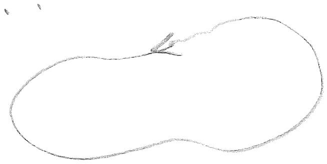
$$\operatorname{Curl} \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = f_{yx} - f_{xy} = 0.$$

In essence, it is the same theorem as before and goes only one way: if $\operatorname{curl}\vec{F}(x,y) \not\equiv 0$, then $\vec{F}(x,y)$ is not conservative. What if $\operatorname{curl}\vec{F}(x,y) \equiv 0$, can we conclude that $\vec{F}(x,y)$ is conservative? It does not follow from what we have done so far. Is it true? Yes, under some additional assumptions. The corresponding theorem is called the "Curl Test". Before we talk about the Curl Test, we need another very important theorem: Green's Theorem.

A couple of "definitions":

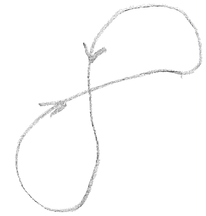
1) A <u>closed curve C</u> in the xy-plane is called <u>simple</u> if it does not cross itself.

E.g.:



C

simple



C

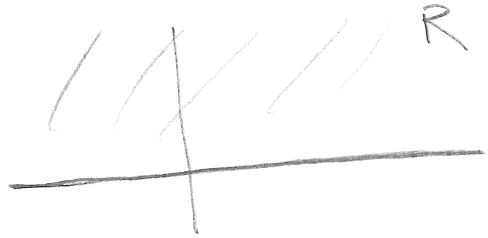
not simple

2) A <u>region</u> R in the xy-plane is called <u>open</u> if it does not contain its boundary or, in other words, if a point P is in R, then there exists a disk around P that is entirely contained in R.

E.g.:

R

R: $y > 0$ open

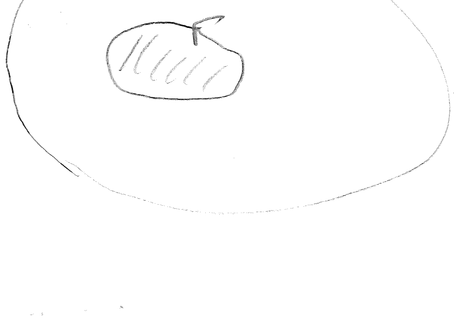


R

R: $y \geq 0$ not open.

3) Let's say that a region R in the xy-plane "has no holes" if every closed curve contained in R encircles only points of R:

R

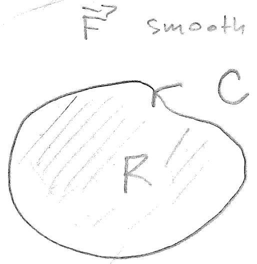


No holes

R

A hole

Th (Green's Theorem): Let C be a piecewise smooth simple closed curve oriented ccw. Let R be the open region enclosed by C. Let $\vec{F}(x,y) = F_1(x,y)\vec{i} + F_2(x,y)\vec{j}$ be a smooth vector field on an open region containing C and R. Then:



$$\oint_C \vec{F} \cdot d\vec{r} = \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA =$$

$$= \int_R (\text{curl}\, \vec{F})\, dA.$$

The theorem relates the line integral to the double integral. In particular, it provides yet another way to evaluate line integrals. The assumption that $\vec{F}$ is defined on C and everywhere in the region R is of crucial importance. If there are points in R where $\vec{F}$ is not defined, the formula may not hold.

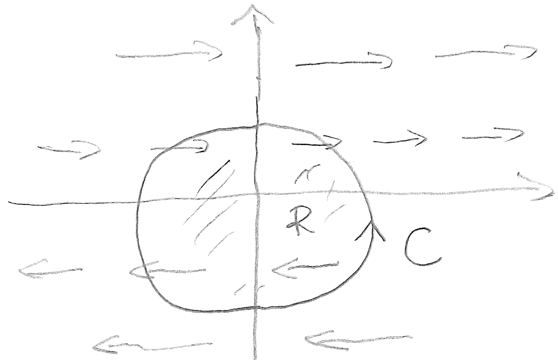
Ex: Let $\vec{F} = y\vec{i}$. Let C be the unit circle oriented ccw.

(a) Sketch $\vec{F}$.

(b) Use Green's Theorem to calculate the circulation:

$$\int_C \vec{F}\, d\vec{r}.$$

(a)



We can tell:

$$\int_C \vec{F} < 0,$$

So $\vec{F}$ is not conservative for sure.

(b) We could parametrize C to calculate the integral. But we can also use Green's Theorem. All assumptions are satisfied. $\text{curl}\,\vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -1$.

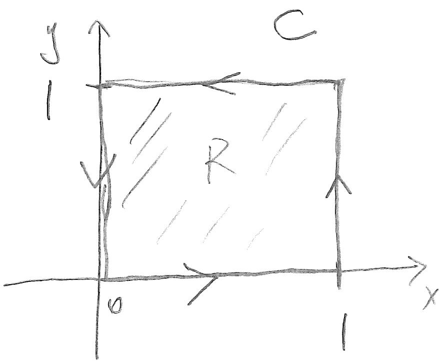
$$\int_C \vec{F}\,d\vec{r} = \int_R (\text{Curl}\,\vec{F})\,dA = \int_R -1\,dA = -\text{Area}(R) = -\pi.$$

Or in polar coordinates:

$$\int_R 1\,dA = \int_0^1 \int_0^{2\pi} -1\;r\,d\theta\,dr = \int_0^1 -2\pi r\,dr = -\pi r^2\Big|_0^1 = \underline{\underline{-\pi}}.$$

Green's theorem is the simplest way.

<u>Ex</u>: Let $\vec{F} = xy\,\vec{j}$, C be the perimeter of the square $0 \le x \le 1$, $0 \le y \le 1$ oriented ccw. Find $\int_C \vec{F}$.



Green's Theorem is applicable.

$\text{curl}\,\vec{F} = y - 0 = y$. So

$$\int_C \vec{F} = \int_R y\,dA = \int_0^1\int_0^1 y\,dx\,dy =$$

$$= \int_0^1 y\,dy = \boxed{\tfrac{1}{2}}$$

The last theorem this semester!

<u>Th. (The Curl Test)</u>. Assume that $\vec{F} = F_1\vec{i} + F_2\vec{j}$ is a vector field with continuous partial derivatives defined on a domain D that has no holes. Assume that

$$\operatorname{curl}\vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0 \qquad \text{in } D.$$

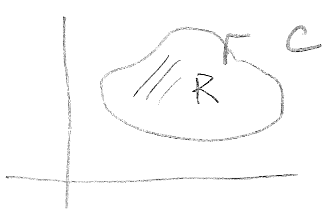
Then $\vec{F}$ is conservative, that is, $\vec{F}$ has a potential function in D.

In other words, as long as we consider $\vec{F}$ on a domain with no curl $\vec{F} = 0$ implies that $\vec{F}$ is conservative, curl $\vec{F} \neq 0$ implies that $\vec{F}$ is not conservative.

The Curl Test follows easily from Green's Theorem. We know that $\vec{F}$ is conservative if and only if

$$\oint_C \vec{F}\, d\vec{r} = 0$$

for any closed path. Suppose curl $\vec{F} = 0$ and take any closed path C. Then by Green's Theorem:



$$\oint_C \vec{F}\, d\vec{r} = \int_R (\operatorname{curl}\vec{F})\, dA = 0.$$

(Provided $\vec{F}$ is defined throughout R. This is where we need the domain of $\vec{F}$ to have no holes.)

So in a domain with no holes:

$\vec{F}$ is conservative $\Longleftrightarrow$ curl $\vec{F} = 0$

Ex: Determine if a given vector field is conservative. If yes, find a potential function.

(a) $\vec{F} = y\vec{i} + y\vec{j}$ (b) $\vec{F} = (x^2+y^2)\vec{i} + 2xy\vec{j}$

(a) curl $\vec{F} = \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = 0 - 1 = -1 \neq 0$. $\vec{F}$ is not conservative. In other words, $\vec{F}$ is not path-independent and it does not have a potential function.

(b) curl $\vec{F} = 2y - 2y = 0$. The domain of $\vec{F}$ is the whole xy-plane. Thus $\vec{F}$ is conservative and has a potential (by the Curl Test).

Let's find a potential function, $f(x,y)$.

$$?f?, \quad f_x\vec{i} + f_y\vec{j} = (x^2+y^2)\vec{i} + 2xy\vec{j}$$

$$f_x = x^2+y^2, \quad f_y = 2xy$$

$$f(x,y) = \tfrac{1}{3}x^3 + y^2x + C(y) \rightsquigarrow f_y = 2xy + C'(y) \rightsquigarrow C(y) = 0$$

Thus:

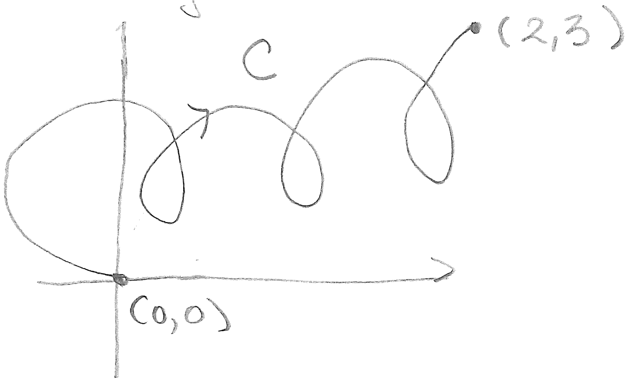
$$f(x,y) = \frac{1}{3}x^3 + y^2 x$$

is a potential function for $\vec{F}$. Check:

$$\nabla f = (x^2+y^2)\vec{i} + 2yx\,\vec{j} = \vec{F} \quad \checkmark$$

Ex: Let $\vec{F}$ be as in (6). Find $\int_C \vec{F}\,d\vec{r}$

along C:



$$\int_C \vec{F}\,d\vec{r} = \int_{(0,0)}^{(2,3)} \vec{F} = f(2,3) - f(0,0) =$$

$$= \frac{8}{3} + 18.$$

The assumption in the Curl Test that we consider $\vec{F}$ in a domain with no holes is easy to overlook and it is crucial.

Ex: Consider

$$\vec{F}(x,y) = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}.$$

Is $\vec{F}$ conservative in the xy-plane? Apply the Curl Test:

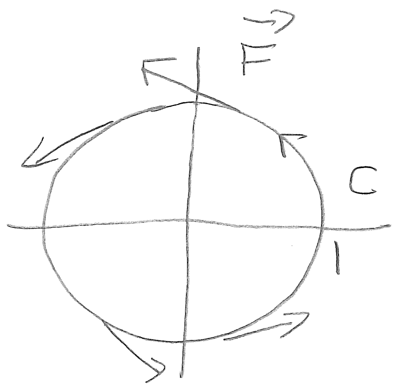
Quotient rule

$$\operatorname{curl}\vec{F} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y)^2} -$$

$$- \frac{y^2-x^2}{(x^2+y^2)^2} = 0$$

$$\frac{1(x^2+y^2) - x(2x)}{(x^2+y^2)^2}$$

curl $\vec{F} = 0$. Thus you might conclude that $\vec{F}$ is conservative in the xy-plane. Is it the case?



$$\int_C \vec{F}\, d\vec{r} > 0 \quad ????$$

$\vec{F}(x,y) = -y\vec{i} + x\vec{j}$ on the unit circle as $x^2+y^2=1$.

Where is a mistake? The Curl Test does not apply as $\vec{F}(x,y)$ is not defined at $(0,0)$!