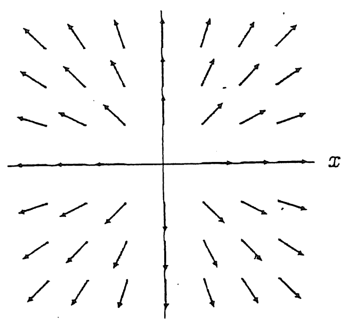
R: Quiz: 17.2, 17.3 in class

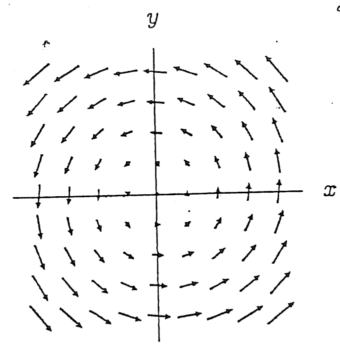
April 14, 16

Ex: Find a possible formula for:

(a) $\vec{F}(x,y)$



(b) $\vec{G}(x,y)$



(a) $\vec{F}(x,y)$ points directly away from the origin, so

$\vec{F}(x,y) \parallel x\vec{i} + y\vec{j}$,

$\|\vec{F}(x,y)\| = \text{const}$. So for example $\vec{F}(x,y)$ may be

$$\vec{F}(x,y) = \frac{x}{\sqrt{x^2+y^2}}\vec{i} + \frac{y}{\sqrt{x^2+y^2}}\vec{j}$$

$\|\vec{F}\| \equiv 1$. In terms of $\vec{r} = x\vec{i} + y\vec{j}$?

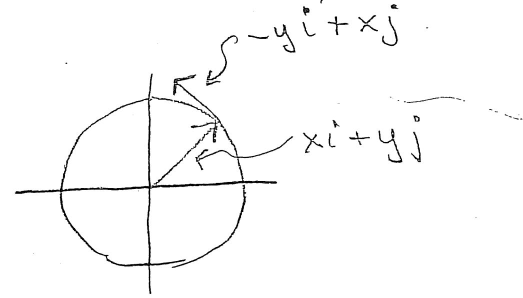
$$\vec{F}(\vec{r}) = \frac{\vec{r}}{\|\vec{r}\|}.$$

$\vec{F}(x,y)$ is not defined at $(0,0)$.

[A gradient field for $f(x,y) = \sqrt{x^2+y^2}$.]

(b) $\vec{G}(x,y)$ seems tangent to circles centered at the origin, magnitude, $\|\vec{G}(x,y)\|$ constant on each circle, increases as the radius of the circle increases. $\vec{G}(x,y)$ points ccw.

$$\vec{G}(x,y) \perp x\vec{i} + y\vec{j}$$



Try: $\vec{G}(x,y) = -y\vec{i} + x\vec{j}$

$\|\vec{G}(x,y)\| = \sqrt{x^2+y^2}$, points the right way.

Is $\vec{G}$ a gradient field?

? $f(x,y)$? $f_x = -y$, $f_y = x$

$$f = -yx + C(y) \to f_y = -x + C'(y) \neq x.$$

No. We can try the test:

$$\frac{\partial F_1}{\partial y} \overset{?}{=} \frac{\partial F_2}{\partial x} \quad ? \qquad F_1 = -y, \; F_2 = x$$

$$\frac{\partial F_1}{\partial y} = -1, \quad \frac{\partial F_2}{\partial x} = 1$$

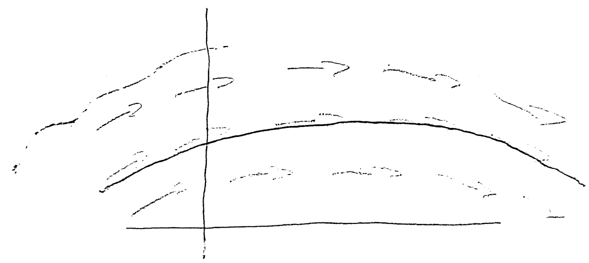
No. Not a gradient field.

17.4 The Flow of a Vector Field

Let's have a vector field, say on the xy-plane:

$$\vec{F}(x,y) = \vec{F}(\vec{r}). \qquad (\vec{r} = x\vec{i} + y\vec{j}.)$$

Suppose for a moment that $\vec{F}(\vec{r})$ is a velocity field, that is shows the velocity of a current, for example, in a body of water.



It is very natural to ask: suppose we put a particle at some point of the plane, what path will the particle follow assuming that its velocity is at each position (x,y) given by $\vec{F}(x,y)$? This path is called a **flow line** of the vector field.

Def: **A flow line** of a vector field $\vec{F}(\vec{r})$ is a curve $\vec{r}(t)$ whose velocity vector $\vec{v}(t)$ satisfies

$$\vec{v}(t) = \vec{r}\,'(t) = \vec{F}(\vec{r}(t)).$$

The flow of a vector field is the family of all its flow lines.

How to find the flow of a given vector field? Let's resolve everything into components:

$$\vec{F}(x,y) = F_1(x,y)\vec{i} + F_2(x,y)\vec{j}$$

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$$

$$\vec{v}(t) = \vec{r}\,'(t) = x'(t)\vec{i} + y'(t)\vec{j}$$

So $\vec{r}\,'(t) = \vec{F}(\vec{r}(t)) = F_1(x(t),y(t))\vec{i} + F_2(x(t),y(t))\vec{j}$ means:

$$x'(t) = F_1(x(t),y(t)), \quad y'(t) = F_2(x(t),y(t))$$

A system of DE's. Not always easy to solve.

Ex: Find the flow of

$$\vec{F}(x,y) = 2x\vec{i} + 2y\vec{j}$$

We set up the equations for $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$:

$$\begin{cases} x'(t) = F_1(x(t), y(t)) = 2x(t) \\ y'(t) = F_2(x(t), y(t)) = 2y(t) \end{cases}$$

Simple in this case:

$$\begin{cases} x'(t) = 2x(t) \\ y'(t) = 2y(t) \end{cases}$$

$$x(t) = C_1 e^{2t}, \quad y(t) = C_2 e^{2t}$$

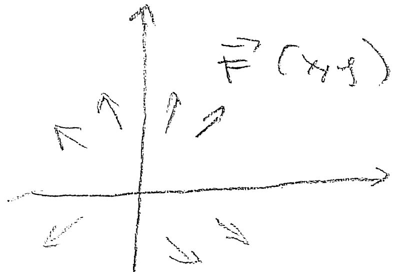
If if have a starting point $x(0) = x_0$, $y(0) = y_0$, we get the flow line:

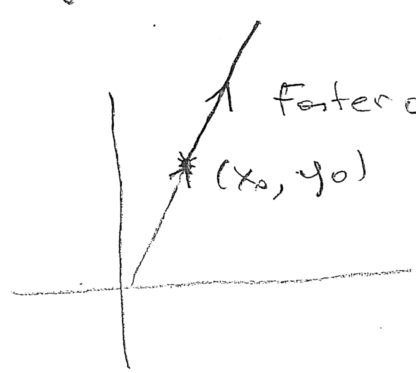
$$x(t) = x_0 e^{2t}, \quad y(t) = y_0 e^{2t}$$

or in vector form:

Half-line as $e^{2t} > 0$.

$$\vec{r}(t) = x_0 e^{2t}\vec{i} + y_0 e^{2t}\vec{j} = e^{2t}(x_0\vec{i} + y_0\vec{j})$$





The system of ODE's was so simple as $x(t), y(t)$ were separated.

Ex: Find the flow for:

$$\vec{F}(x,y) = -y\vec{i} + x\vec{j}$$

We get:

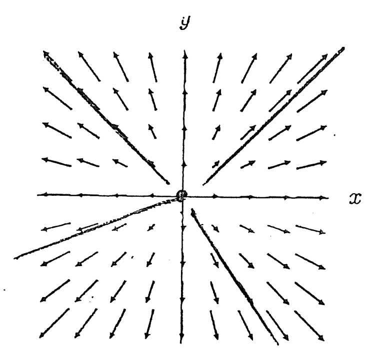
$$\begin{cases} x'(t) = -y(t) \\ y'(t) = x(t) \end{cases}$$

Guessing:

$$x(t) = a\cos(t), \quad y(t) = a\sin(t)$$

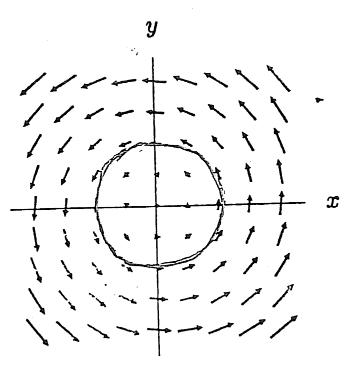
for any constant a.

Picture for the previous example $\vec{F}(x,y) = 2x\vec{i} + 2y\vec{j}$:



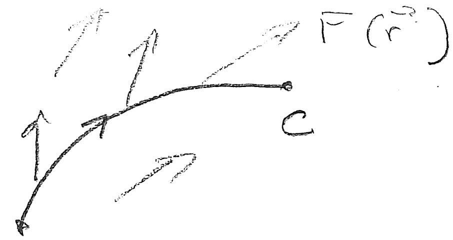
At $(0,0)$, $\vec{F}(0,0) = \vec{0}$ - - a stationary point.

Picture for $\vec{F}(x,y) = -y\vec{i} + x\vec{j}$:



18.1 Line Integrals

Let a vector field $\vec{F}(\vec{r})$ and an oriented curve C on the xy-plane we given. "Oriented" means that we have a direction of motion on C. How do we define the line integral:



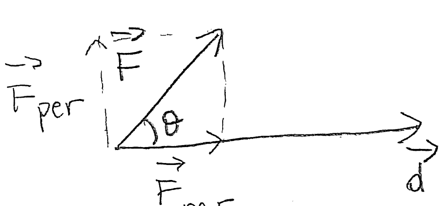
$$\int_C \vec{F}(\vec{r})\,d\vec{r} \quad ?$$

We define it in such a way that if $\vec{F}(\vec{r})$ is a force-field, then the integral represents the work done be the force as the body moves along C. What definition will capture that?

Recall the basic formula:

$$\text{Work} = \text{Force} \times \text{Distance}$$

which is valid provided an object move along a directected segment in the exact diretion of the force. What if the displacement and the force are not parallel:



$$\text{Work} = \|\vec{F}_{par}\| \cdot \|\vec{d}\,\|$$

The dot product.

$$\|\vec{F}_{per}\| = \|\vec{F}\| \cdot \cos\theta$$

So

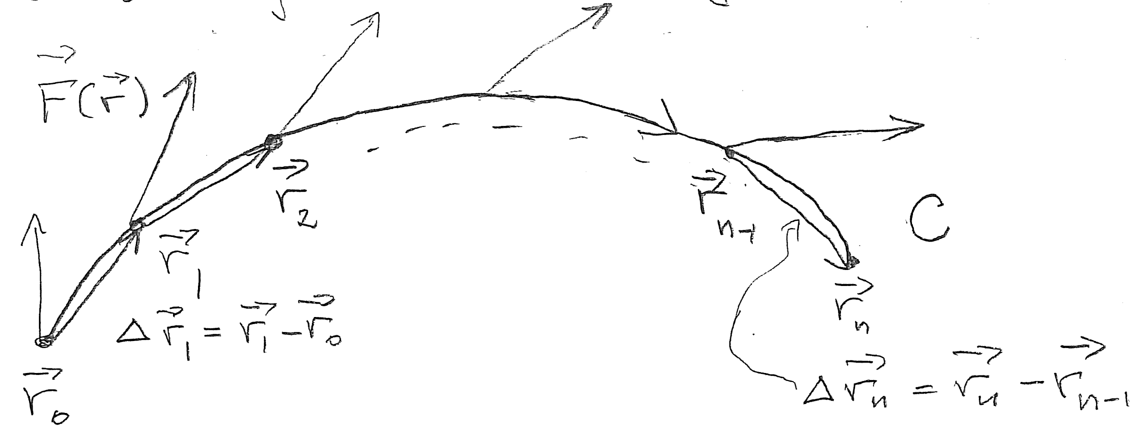
$$\underline{\text{Work}} = \|\vec{F}\| \cdot \|\vec{d}\,\| \cdot \cos\theta = \underline{\vec{F} \cdot \vec{d}}$$

Work is the dot product of the force and displacement.

Now we will define

$$\int_C \vec{F}(\vec{r})\,d\vec{r}$$

in such a way that the integral represent work.



We divide C into very small pieces by choosing points $\vec{r}_0, \vec{r}_1, \dots, \vec{r}_n$ along C. If C is smooth each piece is almost a segment so the displacement from $\vec{r}_{i-1}$ to $\vec{r}_i$ along C practically coincides with the displacement along the vector $\Delta\vec{r}_i = \vec{r}_i - \vec{r}_{i-1}$. $\vec{F}(\vec{r})$ changes along a small piece very little as well (if $\vec{F}(\vec{r})$ is continuous). Thus:

$$\text{Work over the } i\text{-th piece} \cong \vec{F}(\vec{r}_i)\cdot\Delta\vec{r}_i$$

$$\text{Total work} \cong \sum_i \vec{F}(\vec{r}_i)\,\Delta\vec{r}_i$$

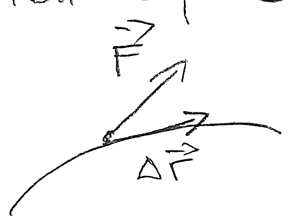
We get better and better approximation as $\|\Delta\vec{r}_i\| \to 0$:

$$\text{Total work} = \lim_{\|\Delta\vec{r}_i\|\to 0} \sum_i \vec{F}(\vec{r}_i)\cdot\Delta\vec{r}_i \triangleq \int_C F(\vec{r})\,d\vec{r}$$

Under proper regularity assumptions about $\vec{F}$ and C (C smooth, $\vec{F}$ continuous) the limit exists and does not depend on the choice of $\vec{r_i}$. This limit is the line integral:

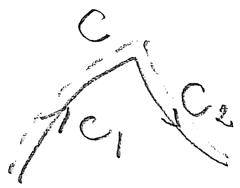
$$\int_C \vec{F}(\vec{r})d\vec{r} = \lim_{\|\Delta\vec{r_i}\|\to 0} \sum_i \vec{F}(\vec{r_i})\cdot\Delta\vec{r_i}$$

Physically the integral represents work. Mathematically it represents "how much" the vector field $\vec{F}$ "goes in the direction" of C, "along C":



$\vec{F}\cdot\Delta\vec{r}$ $\qquad$ $\vec{F}\perp C \to \int_C \vec{F} = 0$

A few additional definitions and properties of the line integral:

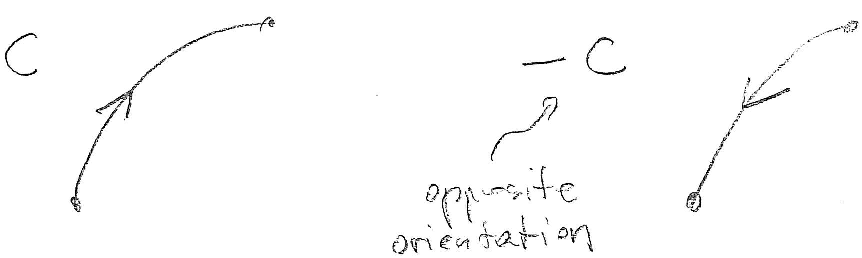


$C = C_1 + C_2$

Notation

The definition of $\int_C \vec{F} d\vec{r}$ can be extended to piecewise smooth path.

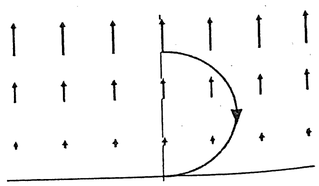
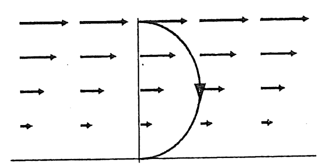
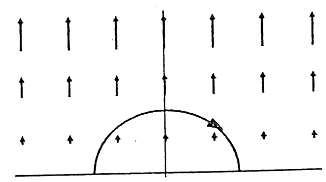
$$\int_C \vec{F}(\vec{r})d\vec{r} = \int_{C_1} \vec{F}(\vec{r})d\vec{r} + \int_{C_2} F(\vec{r})d\vec{r}$$



For a scalar constant λ, vector fields $\vec{F}$ and $\vec{G}$, and oriented curves C, C_1, and C_2

1. $\int_C \lambda\vec{F}\cdot d\vec{r} = \lambda\int_C \vec{F}\cdot d\vec{r}$.
2. $\int_C (\vec{F}+\vec{G})\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r} + \int_C \vec{G}\cdot d\vec{r}$.
3. $\int_{-C} \vec{F}\cdot d\vec{r} = -\int_C \vec{F}\cdot d\vec{r}$.
4. $\int_{C_1+C_2} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$.

Ex: Is $\int_C \vec{F}(\vec{r})d\vec{r}$ positive, negative or 0? C and $\vec{F}$ are depicted below.

(a)  (b)  (c) 

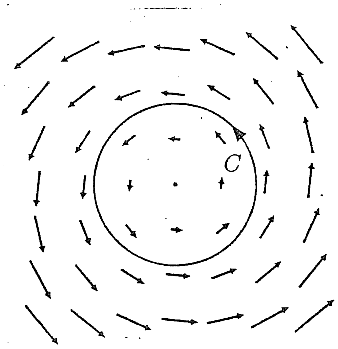
Negative. Positive. 0.

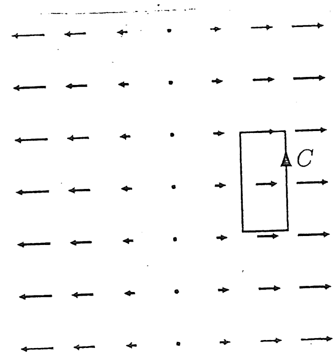
Def: If C is a closed oriented curve, then

$$\int_C \vec{F}(\vec{r})d\vec{r} = \oint_C \vec{F}(\vec{r})d\vec{r}$$

is called the circulation of $\vec{F}$ around C.

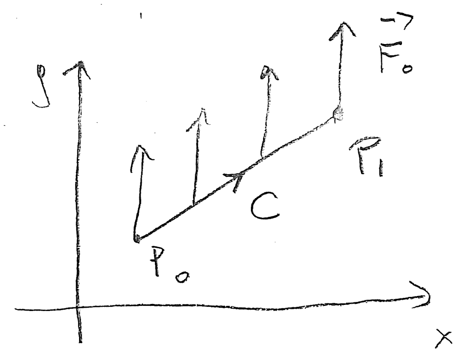
Ex: Find the sign of the circulation of the vector fields around the indicated paths.

(a) 

(b) 

Positive. Zero.

<u>Remark</u>: Let $\vec{F}(\vec{r}) = \vec{F_0}$ be constant. Let C be an oriented straight line segment from P_0 to P_1.



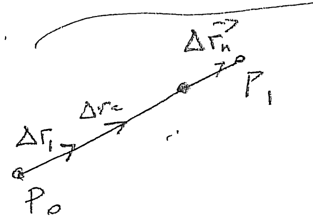
Then:

$$\int_C \vec{F}(\vec{r})\,d\vec{r} = \vec{F_0} \cdot \overrightarrow{P_0P_1}$$

Follows easily from the definition of the line integral:

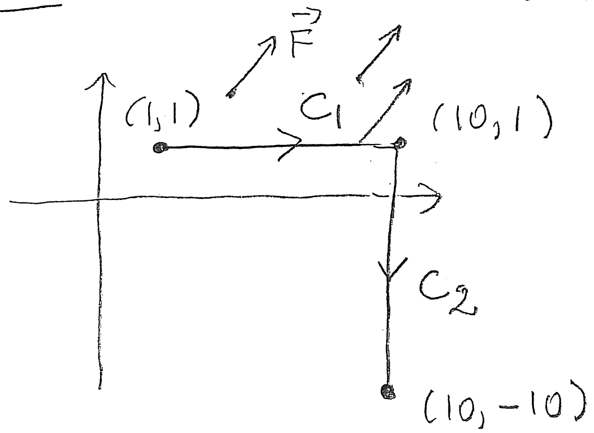
$$\int_C \vec{F}\,d\vec{r} = \lim_{\|\Delta\vec{r_i}\| \to 0} \sum_i \vec{F}(\vec{r_i}) \cdot \Delta\vec{r_i} = \lim_{\|\Delta\vec{r_i}\| \to 0} \sum_i \vec{F_0} \cdot \Delta\vec{r_i} =$$

$$= \lim_{\|\Delta\vec{r_i}\| \to 0} \vec{F_0} \cdot \left(\sum_i \Delta\vec{r_i}\right) = \lim_{\|\Delta\vec{r_i}\| \to 0} \vec{F_0} \cdot \overrightarrow{P_0P_1} = \vec{F_0} \cdot \overrightarrow{P_0P_1}$$



$$\sum_i \Delta\vec{r_i} = \overrightarrow{P_0P_1}$$

<u>Ex</u>: Let $\vec{F} = \vec{i} + 2\vec{j}$, C be the following path:



$$C = C_1 + C_2$$

$$\int_C \vec{F}\,d\vec{r} = \int_{C_1} \vec{F}\,d\vec{r} + \int_{C_2} \vec{F}\,d\vec{r} =$$

$$= (\vec{i} + 2\vec{j}) \cdot (9\vec{i}) +$$

$$+ (\vec{i} + 2\vec{j}) \cdot (-11\vec{j}) =$$

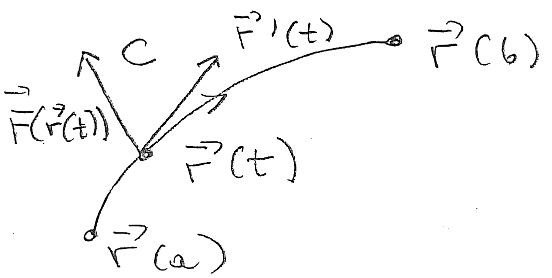
$$= 9 - 22 = \underline{-13}$$

In some simple cases we can find the line integral from the Remark or from the definition. What do we do in general? Parametrize C.

If $\vec{r}(t)$, for $a \le t \le b$, is a smooth parameterization of an oriented curve C and $\vec{F}$ is a vector field which is continuous on C, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt.$$

In words: To compute the line integral of $\vec{F}$ over C, take the dot product of $\vec{F}$ evaluated on C with the velocity vector, $\vec{r}\,'(t)$, of the parameterization of C, then integrate along the curve.



$$\int_C \vec{F}\, d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

dot product.

In terms of coordinates:

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}, \quad t \text{ in } [a,b],$$

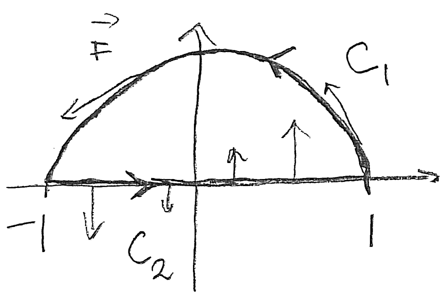
$$\vec{F}(\vec{r}(t)) = F_1(x(t), y(t))\vec{i} + F_2(x(t), y(t))\vec{j}$$

$$\int_C \vec{F}\, d\vec{r} = \int_a^b \left(F_1(x(t), y(t))\vec{i} + F_2(x(t), y(t))\vec{j}\right) \cdot \left(x'(t)\vec{i} + y'(t)\vec{j}\right) dt$$

Ex: Find $\int_C \vec{F}(\vec{r})\,d\vec{r}$ where:

$\vec{F}(x,y) = -y\vec{i} + x\vec{j}$,

C is given by $C = C_1 + C_2$:



C_1 - the upper hemicircle of the unit circle oriented ccw.

C_2 - the segment $(-1,0), (1,0)$.

$$\int_C \vec{F}\,d\vec{r} = \int_{C_1} \vec{F}\,d\vec{r} + \int_{C_2} \vec{F}\,d\vec{r}$$

Note $\vec{F} \perp C_2$ at each point along C_2. Thus $\int_{C_2} \vec{F} = 0$.

We can check by parametrization:

C_2: $\quad x(t) = t, \quad y(t) = 0, \qquad t$ in $[-1,1]$.

Thus:

$$\int_{C_2} \vec{F}(\vec{r})\,d\vec{r} = \int_{-1}^{1} F(x(t), y(t))] \cdot (x'(t)\vec{i} + y'(t)\vec{j})\,dt =$$

$$= \int_{-1}^{1} (t\vec{j}) \cdot (\vec{i})\,dt = \int_{-1}^{1} 0\,dt = 0.$$

Now C_1..

C_1: $\quad x(t) = \cos(t), \quad y(t) = \sin(t)$, t in $[0, \pi]$

or, in vector form:

$$\vec{r}(t) = \cos(t)\vec{i} + \sin(t)\vec{j}$$

$$\vec{r}\,'(t) = -\sin(t)\vec{i} + \cos(t)\vec{j}.$$

$$\vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t)) = -\sin(t)\vec{i} + \cos(t)\vec{j},$$

Thus:

$$\int_{C_1} \vec{F}(\vec{r})\,d\vec{r} = \int_0^{\pi} (-\sin(t)\vec{i} + \cos(t)\vec{j}) \cdot (-\sin(t)\vec{i} + \cos(t)\vec{j})\,dt =$$

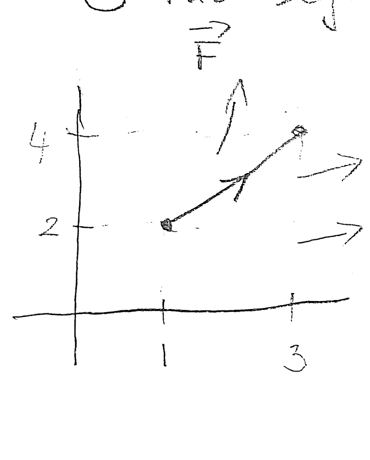
$$= \int_0^{\pi} \underbrace{(\sin^2(t) + \cos^2(t))}_{=1}\,dt = \pi.$$

Thus:

$$\int_C \vec{F}\,d\vec{r} = \pi$$

<u>Ex</u>: Find $\int_C \vec{F}$ where $\vec{F}(\vec{r}) = x^2\vec{i} + y^2\vec{j}$,

C the segment from $(1, 2)$ to $(3, 4)$.



Parametrize C:

$$\vec{w} = \overrightarrow{(1,2)(3,4)} = 2\vec{i} + 2\vec{j}$$

So we could take:

$$x = 1 + 2t, \quad y = 2 + 2t, \quad t \text{ in } [0, 1].$$

We can take instead a simpler parametrization:

$$C: \quad x(t) = 1 + t, \quad y(t) = 2 + t, \quad t \text{ in } [0, 2].$$

We have:

$$\vec{r}(t) = (1+t)\vec{i} + (2+t)\vec{j},$$

$$\vec{r}\,'(t) = \vec{i} + \vec{j}$$

$$\vec{F}(\vec{r}(t)) = (1+t)^2\vec{i} + (2+t)^2\vec{j}.$$

So:

$$\int_C \vec{F}\, d\vec{r} = \int_0^2 \left((1+t)^2\vec{i} + (2+t)^2\vec{j}\right)\cdot(\vec{i} + \vec{j})\, dt =$$

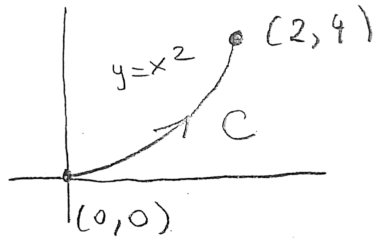
$$= \int_0^2 \left((1+t)^2 + (2+t)^2\right) dt = \int_0^2 (1 + 2t + t^2 + 4 + 4t + t^2)\, dt =$$

$$\int_0^2 (5 + 6t + 2t^2)\, dt = \left(5t + 3t^2 + \tfrac{2}{3}t^3\right)\Big|_0^2 = \frac{82}{3}.$$

Ex: Find $\int_C F(\vec{r})\, d\vec{r}$ where

$$\vec{F}(x,y) = -y \sin x\, \vec{i} + \cos x\, \vec{j},$$

C is the piece of the parabola $y = x^2$ from $(0,0)$ to $(2,4)$.



$C: \; x(t) = t \;, \; y(t) = t^2,$

t in $[0,2]$.

$$\vec{r}(t) = t\vec{i} + t^2\vec{j} \;, \quad \vec{r}\,'(t) = \vec{i} + 2t\vec{j}$$

$$\vec{F}(\vec{r}(t)) = -t^2 \sin(t)\,\vec{i} + \cos(t)\,\vec{j}$$

$$\int_C \vec{F}\,d\vec{r} = \int_0^2 \left(-t^2 \sin(t) + 2t\cos(t)\right) dt =$$

$-\sin(t) = (\cos(t))'$ $\quad 2t = (t^2)'$

$$= t^2 \cos(t)\Big|_0^2 = \underline{4\cos(2)} \, .$$

You can use your graphing calculator.