MTH 536 S07 Homework 2 – Solutions

1) As $f_n \xrightarrow{\|\cdot\|_\infty} f$ by Th 31.1, we have for some $E \subseteq [a,b]$, $m(E)=0$:

$$f_n \rightrightarrows f \quad \text{on} \quad [a,b] \sim E.$$

The latter gives

$$|f_n - f| \rightrightarrows 0 \quad \text{on} \quad [a,b] \sim E,$$

which in turns implies for $1 \le p < +\infty$:

$$|f_n - f|^p \rightrightarrows 0 \quad \text{on } [a,b] \sim E. \quad (1)$$

(For $p = +\infty$, there is nothing to prove.) To prove that $f_n \xrightarrow{\|\cdot\|_p} f$, it suffies to show that

$$\left(\int_{[a,b]} |f_n - f|^p\right) \to 0 \quad \text{as } n \to +\infty. \quad (2)$$

Let $\varepsilon > 0$ be given. Let $N \in \mathbb{N}$ be such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2(b-a)} \quad \text{for } x \in [a,b] \sim E, \; n \ge N. \quad (3)$$

(Such $N$ exists by (1).). From additivity of the integral and (3), we have for all $n \ge N$:

$$\int_{[a,b]} |f_n - f|^p = \int_{[a,b]\sim E} |f_n - f|^p + \int_E |f_n - f|^p \le$$

$$\le \frac{\varepsilon}{2} + 0 < \varepsilon.$$

Thus, (2).

2) Let $\delta > 0$ be fixed. For all $n = 1, 2, \ldots$, consider

$$A_n^\delta = \{x \in [a,b] : |f_n(x) - f(x)| \ge \delta\}.$$

To prove that $f_n \xrightarrow{m} f$ it suffices to show that

$$m(A_n^\delta) \to 0 \quad \text{as} \quad n \to +\infty. \quad (4)$$

Since $f_n \xrightarrow{\|\cdot\|_1} f$, we have

$$\left(\int_{[a,b]} |f_n - f|\right) \to 0 \quad \text{as } n \to +\infty. \quad (5)$$

Observe that for every $n = 1, 2, \ldots$:

$$\int_{[a,b]} |f_n - f| = \int_{[a,b] \sim A_n^\delta} |f_n - f| + \int_{A_n^\delta} |f_n - f| \geq \int_{A_n^\delta} |f_n - f| \geq$$

$$\geq \delta\, m(A_n^\delta). \ (6)$$

As $\delta$ is fixed, (6) and (5) imply (4).

3) We know that $\mathcal{A}_c$ is a $\sigma$-algebra; we have to prove that $\mu$ is a measure. The conditions (i), (ii) of Def 32.1 are trivially satisfied. To prove (iii), take a disjoint family $\langle E_n \rangle_{n=1}^{\infty}$, $E_n \in \mathcal{A}_c$.
If all elements $E_n$ are countable, so is $\bigcup_{n=1}^{\infty} E_n$ and:

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = 0 = \sum_{n=1}^{\infty} \mu(E_n).$$

Suppose at least one element, $E_{n_0}$, is uncountable; that is, $\sim E_{n_0}$ is countable. Since $\langle E_n \rangle_{n=1}^{\infty}$ is disjoint, $E_n \subseteq \sim E_{n_0}$ for all $n \neq n_0$. Thus, all $E_n$ for $n \neq n_0$ are countable and:

$$\sum_{n=1}^{\infty} \mu(E_n) = \mu(E_{n_0}) = 1.$$

As $\sim \bigcup_{n=1}^{\infty} E_n = \bigcap_{n=1}^{\infty} \sim E_n \subseteq \sim E_{n_0}$, $\sim \bigcup_{n=1}^{\infty} E_n$ is countable. Thus, $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = 1$ and (iii) is proved.

4) The proof that $\mathcal{M}|_{[a,b]}$ is a $\sigma$-algebra is very easy and left for the reader. We prove that $\mu_f$ is a measure using Def 32.1 and properties of the integral. (i) is trivial. (ii) follows as $f$ is nonnegative. To prove (iii), we use H9 \#4 from last semester.