MIM536 S07 Homework 1- Solutions

1) We have to show that ρ satisfies (m1)-(m4) of Def 26.1. As $\|\cdot\|$ satisfies (n1)-(n4) of Def 25.2, (m1) follows from (n1). (m2) follows from (n2). To prove (m3), observe that for any $v, w \in V$:

$$\rho(v,w) = \|v-w\| = \|(-1)(w-v)\|$$

From (n4):

$$\rho(v,w) = |(-1)|\,\|w-v\| = \|w-v\| = \rho(w,v).$$

To prove the triangle inequality, take $w, v, z \in V$. We have:

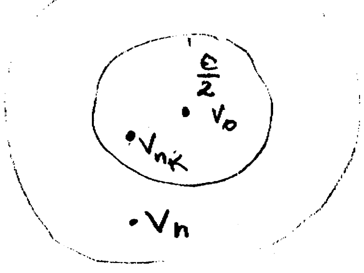
$$\rho(w,v) = \|w-v\| = \|w-z+z-v\| \leq \|w-z\| + \|z-v\| = \rho(w,z)+\rho(z,v),$$

from subadditivity of $\|\cdot\|$. Thus (m4).

2) Let $\langle v_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence from V. Assume that a subsequence $\langle v_{n_k} \rangle_{k=1}^{\infty}$ of $\langle v_n \rangle_{n=1}^{\infty}$ satisfies for a $v_0 \in V$:

$$v_{n_k} \xrightarrow{\|\cdot\|} v_0 \quad \text{as } k \to +\infty. \quad (1)$$

We shall show that



$$v_n \longrightarrow v_0 \quad \text{as } n \to +\infty. \quad (2)$$

Take $\varepsilon > 0$. Let $N \in \mathbb{N}$ be such that

$$\|v_n - v_m\| < \frac{\varepsilon}{2} \quad \text{for } m, n \geq N. \quad (3)$$

(Such N exists as $\langle v_n \rangle$ is Cauchy.) Take $K \in \mathbb{N}$ such that

$$n_K \geq N \quad \text{and} \quad \|v_{n_K} - v_0\| < \frac{\varepsilon}{2}. \quad (4)$$

(Such K exists by (1) and the definition of a subsequence.) Then by (3), (4) we have that for all $n \geq N$:

$$\|v_n - v_0\| \leq \|v_n - v_{n_K}\| + \|v_{n_K} - v_0\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, (2).

3) Follows from the Lebesgue Dominated Convergence Theorem. The problem is the same as MTH 535 H 9 #6.

4) Observe that $\frac{q}{p}$ and $\frac{q}{q-p}$ are conjugate exponents. We shall apply Hölder's inequality to the functions $|f|^p$ and 1. Hölder's inequality gives:

$$\| |f|^p \cdot 1 \|_1 \le \| |f|^p \|_{\frac{q}{p}} \cdot \| 1 \|_{\frac{q}{q-p}} .$$

That is:

$$\int_{[a,b]} |f|^p \le \left(\int_{[a,b]} (|f|^p)^{\frac{q}{p}} \right)^{\frac{p}{q}} \cdot \left(\int_{[a,b]} 1 \right)^{\frac{q-p}{q}}$$

Simplifying the obvious and taking the power of $\frac{1}{p}$ of both sides we obtain:

$$\left(\int_{[a,b]} |f|^p \right)^{\frac{1}{p}} \le \left(\int_{[a,b]} |f|^q \right)^{\frac{1}{q}} \cdot \left(\int_{[a,b]} 1 \right)^{\frac{q-p}{pq}}$$

As $\int_{[a,b]} 1 = b-a$ and $\frac{q-p}{pq} = \frac{1}{p} - \frac{1}{q}$, we obtain:

$$\| f \|_p \le \| f \|_q \cdot (b-a)^{\frac{1}{p}-\frac{1}{q}} .$$

5) If $\| f \|_\infty = +\infty$ or $\| g \|_\infty = +\infty$, the inequality

$$\| f+g \|_\infty \le \| f \|_\infty + \| g \|_\infty \qquad (3)$$

holds. Assume $f, g \in \mathcal{L}^\infty([a,b])$, that is, they are essentially bounded. Let M be any a.e. upper bound for $|f|$, N any a.e. upper bound for $|g|$, $M, N \in \mathbb{R}$. Denote

$$A = \{x \in [a,b] : |f| > M\}, \quad B = \{x \in [a,b] : |g| > N\}.$$

Then $m(A) = m(B) = 0 = m(A \cup B)$. Observe that

$$|f(x) + g(x)| \le |f(x)| + |g(x)| \le M+N \text{ for } x \in [a,b] \setminus (A \cup B).$$

Hence, $M+N$ is an a.e. bound for $|f+g|$ which implies

$$\|f+g\|_\infty \leq M+N. \qquad (4)$$

As (4) holds for any a.e. upper bound M for $|f|$, it holds for the infimum of such upper bounds; $\|f\|_\infty$. That is:

$$\|f+g\|_\infty \leq \|f\|_\infty + N.$$

Similarly, as N was an arbitrary a.e. upper bound for $|g|$ we obtain:

$$\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty.$$

6) Follows easily from the definition of $\|g\|_\infty$ and monotonicity of the integral:

$$\int_{[a,b]} |f\cdot g| = \int_{[a,b]} (|f|\cdot|g|) \leq \int_{[a,b]} (|f|\cdot \operatorname{ess\,sup}|g|) = \int_{[a,b]} (|f|\,\|g\|_\infty) =$$

$$= \|g\|_\infty \cdot \int_{[a,b]} |f| = \|f\|_1 \|g\|_\infty.$$