MTH 436 Homework 6 – Solutions

1) Assume for simplicity that $I = [a,b]$. Since f_n' are continuous in $[a,b]$ and $f_n' \rightrightarrows g$ on $[a,b]$, g is continuous in $[a,b]$ by Th 37.1. f_n is an antiderivative of f_n' in $[a,b]$, hence, by the 1st FTC:

$$f_n(x) - f_n(a) = \int_a^x f_n'(t)\,dt \quad , \quad x \in [a,b]. \quad (1)$$

Fix an $x \in [a,b]$. By Th 38.1 we have:

$$\int_a^x f_n'(t)\,dt \to \int_a^x g(t)\,dt. \qquad (2)$$

Since $f_n(x) - f_n(a) \to f(x) - f(a)$ as $n \to +\infty$, (1) and (2) give:

$$f(x) - f(a) = \int_a^x g(t)\,dt. \qquad (3)$$

Since x was arbitrary, (3) holds for all $x \in [a,b]$. As g is continuous in $[a,b]$ by the 2nd FTC and (3):

$$\frac{d}{dx}\left[\int_a^x g(t)\,dt\right] = \frac{d}{dx}\left[f(x) - f(a)\right] = \frac{d}{dx}\left[f(x)\right] = g(x).$$

Thus, f is differentiable and $f' = g$. (If $x=a$ or $x=b$, the appropriate one-sided derivatives are used.)

If I is an arbitrary interval, we take any point $c \in I$ and use Corollary 35.1. Instead of (1) we have:

$$f_n(x) - f_n(c) = \int_c^x f_n'(t)\,dt \quad \text{for all } x \in I.$$

The rest of the proof goes the same way.

3) Let $D(x)$ be the Dirichlet function on $[0,1]$. Define $f_n = \frac{1}{n} D$ for $n = 1, 2, \dots$. Each f_n is discontinuous everywhere, yet $f_n \rightrightarrows f$ on $[0,1]$ where $f \equiv 0$ is continuous.

7) $\lim_{n\to+\infty} \sin\left(\frac{n\pi}{2}\right)$ does not exist. Hence, $\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right)$ diverges by Th 40.3. To see that the limit does not exist, take two subsequences, one corresponding to $n_k = 4k$, the other to $n_p = 4p+1$. We have

$$\sin\frac{4k\pi}{2} = \sin(2k\pi) = 0 \longrightarrow 0 \quad \text{as } k\to+\infty$$

$$\sin\left(\frac{(4p+1)\pi}{2}\right) = \sin\left(2p\pi + \frac{\pi}{2}\right) = 1 \longrightarrow 1 \text{ as } p \to +\infty.$$

8) $0 < \frac{2^n}{3^n+1} < \left(\frac{2}{3}\right)^n$. The series $\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$ converges a geometric series with ratio $r = \frac{2}{3} < 1$. Hence, $\sum_{n=0}^{\infty} \frac{2^n}{3^n+1}$ Converges by the SCT.