MTH 436 Homework 5 - Solutions

5.1 Consider f_n on $[0, +\infty)$ defined as

$$f_n(x) = \frac{x+1}{2+nx}$$

for $n = 1, 2, \ldots$.
a. Find the pointwise limit f of $\{f_n\}_{n=1}^{\infty}$ on $[0, +\infty)$.
b. Does this sequence converge uniformly in $[0, +\infty)$?

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a. The pointwise limit f of $\{f_n\}_{n=1}^{\infty}$ is defined as

$$f(x) = \begin{cases} \frac{1}{2}, & x = 0 \\ 0, & x > 0 \end{cases}$$

Indeed, let $x = 0$, $\varepsilon > 0$. Let $N \in \mathbb{N}$ be arbitrary. Then whenever $n \geq N$, we have

$$|f_n(0) - f(0)| = \frac{1}{2} - \frac{1}{2} = 0 < \varepsilon.$$

Let $x > 0$. Let $N \in \mathbb{N}$ be such that

$$\frac{1}{N} < \frac{\varepsilon\,|x|}{|x+1|}.$$

Thus:

$$\frac{1}{|Nx|} < \frac{\varepsilon}{|x+1|}.$$

Then whenever $n \geq N$ we have

$$|f_n(x) - f(x)| = \left|\frac{x+1}{2+nx}\right| < \frac{|x+1|}{|nx|} \leq \frac{|x+1|}{|Nx|} < |x+1|\frac{\varepsilon}{|x+1|} = \varepsilon.$$

Thus, $f_n \to f$.

b. Observe that for $n = 1, 2, \ldots$, f_n is continuous, but f is discontinuous. Therefore $f_n \not\rightrightarrows f$.

2. Consider the sequence $\{f_n\}_{n=1}^{\infty}$, where

$$f_n(x) = \frac{x}{1+n^2x^2}, \qquad \text{for } x \in (-\infty, +\infty), \text{ and for } n = 1, 2, \ldots.$$

Show that the sequence converges uniformly on $(-\infty, +\infty)$ to $f(x) \equiv 0$. (*Hint*: Find the maximum and minimum values of f_n on $(-\infty, +\infty)$.)

Proof. First, note that f_n has a maximum and minimum points at

$$x = \frac{1}{n} \text{ and } x = \frac{-1}{n}. \qquad (1)$$

Further, notice that

$$f_n\left(\frac{1}{n}\right) = \frac{1}{2n} \text{ and } f_n\left(\frac{-1}{n}\right) = \frac{-1}{2n}.$$

It follows that for all $x \in (-\infty, +\infty)$,

$$|f_n(x)| \le \frac{1}{2n} \qquad \text{for } n = 1, 2, \ldots$$

So choose $N \in \mathbb{N}$ such that

$$\frac{1}{2N} < \varepsilon.$$

Then

$$|f_n(x) - f(x)| \le |f_n(x)| + |f(x)| \le \frac{1}{2n} + 0 \le \frac{1}{2N} < \varepsilon \text{ for all } n \ge N.$$

Hence, $f_n \rightrightarrows f$. $\square$

(1) is true but it requires more explanation. Observe that for each n:

$$f_n'(x) = \frac{1+n^2x^2 - 2n^2x^2}{(1+n^2x^2)^2} = \frac{1-n^2x^2}{(1+n^2x^2)}.$$

Thus, f_n has two critical points: $c_1 = -\frac{1}{n}$, $c_2 = \frac{1}{n}$. Since, f_n is odd, let's look at the interval $[0, +\infty)$. We have $f_n'(x) > 0$ for $x \in [0, \frac{1}{n})$, $f_n'(x) <$ for $x \in (\frac{1}{n}, +\infty)$. Hence, f_n has the global maximum in $[0, +\infty)$ at $\frac{1}{n}$. Also, $f_n(x) > 0$ for $x > 0$. Since f_n is odd, f_n has the global minimum in $(-\infty, +\infty)$ at $-\frac{1}{n}$ and the global maximum in $(-\infty, +\infty)$ at $x = \frac{1}{n}$. That is, for all $x \in (-\infty, +\infty)$:

$$f(-\tfrac{1}{n}) \le f(x) \le f(\tfrac{1}{n}).$$

3. For each $n = 1, 2, \ldots$, define g_n on $[0,1]$ as

$$g_n(x) = \begin{cases} \frac{1}{x}, & \frac{1}{n} \le x \le 1 \\ n^2 x, & 0 \le x < \frac{1}{n}. \end{cases}$$

Show that this sequence converges pointwise in $[0,1]$ and find its limit function g. Is the convergence uniform?

The sequence of functions $\{g_n\}_{n=1}^{\infty}$ converges to the limit function

$$g(x) = \begin{cases} 0, & x = 0 \\ \frac{1}{x}, & x \in (0,1]. \end{cases}$$

Proof. Let $x_0 = 0$ and let $\varepsilon > 0$ be given. Then for any $N_0 \in \mathbb{N}$,

$$|g_n(0) - g(0)| = |0 - 0| < \varepsilon \text{ for all } n \ge N_0.$$

Now let $x \in (0,1]$. Choose $N \in \mathbb{N}$ such that $N \ge \frac{1}{x}$. Then

$$|g_n(x) - g(x)| = \left| \frac{1}{x} - \frac{1}{x} \right| = 0 < \varepsilon \text{ for all } n \ge N.$$

□

The convergence, however, is not uniform. The reason is because g_n are continuous functions for all $n \in \mathbb{N}$, but g is not continuous.

4) Assume f_n, f are defined on D, $D \subseteq \mathbb{R}$, $n = 1, 2, \ldots$. Let

$$M_n = \sup\{|f_n(x) - f(x)| : x \in D\} \text{ for } n = 1, 2, \ldots$$

Show that $f_n \rightrightarrows f$ on D if and only if $M_n \to 0$ as $n \to +\infty$.

proof:

" $\Rightarrow$ "

Since $\{f_n\}$ converges uniformly to f (f the limit function of f_n), then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $x \in D$ and $n \geq N$ we have:

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}.$$

Then ~~Let~~ $M_n = \sup\{|f_n(x) - f(x)| : x \in D\} \leq \frac{\varepsilon}{2}$.

Since $M_n \geq 0$, then

$$|M_n| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Therefore, for every $n \geq N$ we have:

$$\{M_n\} \to 0 \text{ as } n \to +\infty.$$

" $\Leftarrow$ "

Let $\{M_n\} \to 0$ as $n \to +\infty$.

Then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|M_n| < \varepsilon.$$

Then ~~Let~~ $M_n = \sup\{|f_n(x) - f(x)|\} < \varepsilon$.

Since

$$|f_n(x) - f(x)| \leq \sup\{|f_n(x) - f(x)|\} < \varepsilon$$

we have for all $x \in I$ and $n \geq N$:

$$|f_n(x) - f(x)| < \varepsilon.$$

Therefore,

$\{f_n\}$ converges uniformly on D.

5) Take $\varepsilon = 1$. Since $f_n \rightrightarrows f$ on $[a,b]$, there exists $N \in \mathbb{N}$ such that

$$|f(x) - f_N(x)| < 1 \quad \text{for all } x \in [a,b].$$

Hence:

$$-1 + f_N(x) < f(x) < 1 + f_N(x) \qquad \text{for all } x \in [a,b]. \quad (2)$$

Let K_N be such that

$$-K_N \leq f_N(x) \leq K_N \qquad \text{for all } x \in [a,b]. \quad (3)$$

(Such K_N exists as f_N is bounded). Combining (2) and (3) we obtain:

$$-1 - K_N < f(x) < 1 + K_N \quad \text{for all } x \in [a,b].$$

Hence, f is bounded on $[a,b]$.

The conclusion does not hold without uniform convergence. In Problem 3, we have sequence of bounded functions converging pointwise to an unbounded function. ∎