6) Let $K = |g(c) - f(c)|$. Assume first that $c \in (a,b)$. We shall prove that

$$g \text{ is integrable on } [a,c] \text{ and } \int_a^c g = \int_a^c f. \quad (1)$$

Let ${}^t\!P \in {}^t\!\mathcal{P}([a,c])$, ${}^t\!P = \{(t_i, [x_{i-1}, x_i]) : i = 1, \dots, n\}$. Observe:

$$|S(f, {}^t\!P) - S(g, {}^t\!P)| = \left| \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) - \sum_{i=1}^n g(t_i)(x_i - x_{i-1}) \right| =$$

$$= |f(t_n)(c - x_{n-1}) - g(t_n)(c - x_{n-1})| \le K \|{}^t\!P\|.$$

Indeed, $f(t_i) = g(t_i)$ for $i = 1, \dots, n-1$. If $t_n \ne c$, $f(t_n) = g(t_n)$ and the latter inequality holds. If $t_n = c$:

$$|f(c)(c - x_{n-1}) - g(c)(c - x_{n-1})| = K|c - x_{n-1}| \le K\|{}^t\!P\|.$$

Hence, for every tagged partition ${}^t\!P \in \mathcal{P}([a,c])$ we have:

$$|S(f, {}^t\!P) - S(g, {}^t\!P)| \le K\|{}^t\!P\|.$$

Take $\varepsilon > 0$. Since f is integrable in $[a,c]$, by Def 28.4 there exists $\delta_1 > 0$ such that:

$$\left| S(f, {}^t\!P) - \int_a^c f \right| < \frac{\varepsilon}{2} \quad \text{whenever } \|{}^t\!P\| < \delta_1.$$

Take $\delta > 0$ such that $\delta < \frac{\varepsilon}{2K}$ and $\delta < \delta_1$. Then, whenever $\|{}^t\!P\| < \delta$:

$$\left| S(g, {}^t\!P) - \int_a^c f \right| < |S(g, {}^t\!P) - S(f, {}^t\!P)| + \left| S(f, {}^t\!P) - \int_a^c f \right| <$$

$$< K\|{}^t\!P\| + \frac{\varepsilon}{2} < K\delta + \frac{\varepsilon}{2} < \varepsilon.$$

Thus, (1) is proved. Similarly, we prove that

$$g \text{ is integrable in } [c,b] \text{ and } \int_c^b g = \int_c^b f. \quad (2).$$

(1) and (2) via Th 32.2 imply that g is integrable in $[a,b]$ and $\int_a^b g = \int_a^b f$. In the case $c = b$, the proof is analogous to the proof of (1). In the case $c = a$, the proof is analogous to the proof of (2). ∎

5) Define g on $[a,b]$ as follows:

$$g(x) = f(x) \text{ for all } x \in (a,b),$$
$$g(a) = f(a+), \quad g(b) = f(b-).$$

Then g is continuous on $[a,b]$ and therefore g is integrable on $[a,b]$. By #6 which we just proved, the function

$$h(x) = g(x), \quad x \in [a,b), \quad h(b) = f(b)$$

is integrable in $[a,b]$ and $\int_a^b h = \int_a^b g$. By #6 again, f is integrable in $[a,b]$ and $\int_a^b f = \int_a^b h = \int_a^b g$.

4) Let $\varepsilon > 0$ be given. We have to show that there exists $N \in \mathbb{N}$ such that

$$|S(f, {}^t\!P_n) - \int_a^b f| < \varepsilon \text{ whenever } n \geq N, \quad (3).$$

Since f is integrable, by Def 28.4, there exists $\delta > 0$ such that

$$|S(f, {}^t\!P) - \int_a^b f| < \varepsilon \text{ whenever } \|{}^t\!P\| < \delta. \quad (4).$$

As $\lim_{n \to +\infty} \|{}^t\!P_n\| = 0$, there exists $N \in \mathbb{N}$ such that

$$\|{}^t\!P_n\| < \delta \text{ for all } n \geq N.$$

From (4), for all $n \geq N$ (3) holds. $\blacksquare$

2) Very easy.

1) We have to show that:

$$\forall_{\varepsilon > 0} \; \exists_{\delta > 0} \; \forall_{x \in [0,1]} \; (0 < x < \delta \Rightarrow f(x) < \varepsilon).$$

Take $\varepsilon > 0$. Take $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \varepsilon$. Take $\delta = \frac{1}{n_0}$.
From the definition of f, whenever $0 < x < \delta$, $f(x) \leq \frac{1}{n_0} < \varepsilon$. $\blacksquare$