1) Let f be integrable in $[a,b]$, Let $k \in \mathbb{R}$ be a constant. Prove
$k f$ is integrable in $[a,b]$ and $\int_a^b kf = k\int_a^b f$

If $k=0$, the equality holds trivially. Assume $k \neq 0$.

Let $\int_a^b f = L$. We want to show that

$$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{{}^tP \in \mathcal{P}[a,b]} \left(\|{}^tP\|<\delta \Rightarrow |S(kf, {}^tP) - kL| < \varepsilon\right)$$

Since f is integrable, there exists $\delta > 0$ such that

$|S(f, {}^tP) - L| < \frac{\varepsilon}{|k|}$ when $\|{}^tP\| < \delta$.

Let ${}^tP = \{(t_i, [x_{i-1}, x_i]) : i = 1, 2, \ldots, n\}$.

Observe: $S(kf, {}^tP) = \sum_{i=1}^{n} kf(t_i)(x_i - x_{i-1}) = k\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$

$= k(S(f, {}^tP))$.

So, $|S(kf, {}^tP) - kL| = |k(S(f, {}^tP)) - kL|$

$= |k(S(f, {}^tP) - L)| = |k||S(f, {}^tP) - L| < |k| \cdot \frac{\varepsilon}{|k|} = \varepsilon$. ■

2) Let $f(x) = c$ for all $x \in [a,b]$ where c is a constant.
Show $\int_a^b f = c(b-a)$

We want to show that for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$|S(f, {}^tP) - c(b-a)| < \varepsilon$ whenever $\|{}^tP\| < \delta$.

take $\varepsilon > 0$. Let ~~δ~~ $\delta > 0$ be arbitrary. Let ${}^tP = \{(t_i, [x_{i-1}, x_i]) : i = 1, 2, \dots, n\}$,

${}^tP \in {}^t\mathcal{P}([a,b])$. Then $S(f, {}^tP) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$

$= \sum_{i=1}^{n} c(x_i - x_{i-1}) = c\sum_{i=1}^{n}(x_i - x_{i-1}) = c(b-a)$.

So, when $\|{}^tP\| < \delta$, we have

$$|S(f, {}^tP) - c(b-a)| = |c(b-a) - c(b-a)| = 0 < \varepsilon.$$

Therefore, $\int_a^b f = c(b-a)$. ■

3) Let f be integrable on $[a,b]$. Then if for some constant $M \in \mathbb{R}$, $|f(x)| \leq M$ for all $x \in [a,b]$, then $|\int_a^b f| \leq M(b-a)$.

Proof: Let $g(x) = M$ for all $x \in [a,b]$

Then we have $|f(x)| \leq g(x)$ for all $x \in [a,b]$.
By Th. 29.1 (c), we have

$$\int_a^b |f(x)| \leq \int_a^b g(x).$$

By HW2 #2, $\int_a^b g(x) = M(b-a)$.

So, $\int_a^b |f(x)| \leq M(b-a)$. But by Thm 31.3,

$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

Therefore, $\left|\int_a^b f\right| \leq M(b-a)$. ■

5) Explain why $\int_{\pi}^{2\pi} \sin^2(x^3)\,dx \approx 3.38$ cannot be correct.

First note that $\sin^2(x^3)$ is integrable in $[\pi, 2\pi]$ as $\sin^2(x^3)$ is continuous.
We see that $f(x) = \sin^2(x^3) \le 1$ for all $x \in [\pi, 2\pi]$. Also, $0 \le \sin^2(x^3)$ for all $x \in [\pi, 2\pi]$.
Therefore, $|f(x)| \le 1$ for all $x \in [\pi, 2\pi]$.
Using Thm 29.1 part (d),

$$\left|\int_{\pi}^{2\pi} f\right| \le (1)(2\pi - \pi) = \pi$$

$$\text{So, } \left|\int_{\pi}^{2\pi} \sin^2(x^3)\right| \le \pi.$$

Therefore, $\int_{\pi}^{2\pi} \sin^2(x^3) \approx 3.38$ cannot be correct as $|3.38| = 3.38 > \pi$.

6) Let f be bounded on $[a,b]$. Prove that $w(|f|, [a,b]) \le w(f, [a,b])$

Using the reverse triangle inequality, we know

$\big|\,|f(t)| - |f(s)|\,\big| \le |f(t) - f(s)|$ for all $t, s \in [a,b]$.

Hence, $\sup\{\big|\,|f(t)| - |f(s)|\,\big| : t, s \in [a,b]\} \le \sup\{|f(t) - f(s)| : t, s \in [a,b]\}$ (1)

By prop 31.1, $w(f, [a,b]) = \sup\{|f(t) - f(s)| : t, s \in [a,b]\}$ and hence $w(|f|, [a,b]) = \sup\{\big|\,|f(t)| - |f(s)|\,\big| : t, s \in [a,b]\}$.

Therefore $w(|f|, [a,b]) \le w(f, [a,b])$ by (1) ▪

7) Let f be bounded on $[a,b]$. Prove

$$w(f,[a,b]) = \sup\{|f(t)-f(s)| : s,t \in [a,b]\} \quad (1)$$

Recall by Def 29.1 that

$$w(f,[a,b]) = \sup\{f(x): x\in[a,b]\} - \inf\{f(x): x\in[a,b]\}$$

Take any $t,s \in [a,b]$. Without loss of generality, assume $f(t) \geq f(s)$. Then we have:

$$|f(t)-f(s)| = f(t)-f(s) \leq \sup\{f(x): x\in[a,b]\} - \inf\{f(x): x\in[a,b]\}.$$

Hence, $w(f,[a,b])$ is an upper bound for $\{|f(t)-f(s)| : s,t\in[a,b]\}$

To prove (1), we use Th 7.3. Note first that if $\omega(f,[a,b]) = 0$, then f is constant in $[a,b]$ and (1) holds trivially. Assume then:

$$\sup\{f(x): x\in[a,b]\} > \inf\{f(x): x\in[a,b]\}.$$

Let $\varepsilon > 0$ be arbitrary such that $\varepsilon < \sup\{f(x) : x\in[a,b]\} - \inf\{f(x): x\in[a,b]\}$. If we prove the condition ~~(2)~~ of Th 7.3 for any such ε, then the condition holds for any $\varepsilon > 0$. By Th 7.3 and Th 7.4, there exists $s_\varepsilon, t_\varepsilon \in [a,b]$ such that

$$\sup\{f(x): x\in[a,b]\} - \frac{\varepsilon}{2} < f(s_\varepsilon) \leq \sup\{f(x): x\in[a,b]\}, \quad (2)$$

~~and~~ $\inf\{f(x): x\in[a,b]\} \leq f(t_\varepsilon) < \inf\{f(x): x\in[a,b]\} + \frac{\varepsilon}{2}$.

Then $f(s_\varepsilon) > f(t_\varepsilon)$. Hence, $|f(s_\varepsilon) - f(t_\varepsilon)| = f(s_\varepsilon) - f(t_\varepsilon)$. Also:

$$-\inf\{f(x): x\in[a,b]\} - \frac{\varepsilon}{2} < -f(t_\varepsilon) \leq -\inf\{f(x): x\in[a,b]\}. \quad (3)$$

Adding (2) and (3), we get

$$\omega(f,[a,b]) - \varepsilon < f(s_\varepsilon) - f(t_\varepsilon) \leq \omega(f,[a,b]).$$

By Th 7.3, (1) holds. $\blacksquare$