MTH 436 S06 Homework 1 - Solutions

1) Let $I = (-\infty, +\infty)$, and let $M = m$. Then prop. 26.2 can be applied as f is differentiable in the open interval I and $f'(x) = m = M \leq M$. So f is Lipschitzian on I, $(-\infty, +\infty)$.

The proof can also be easily obtained from the definition of Lipschitzianity.

2) Let $f(x) = \sqrt{x+1}$. Now, f is continuous on $[-1,1]$, so it is uniformly continuous on $[-1,1]$. In addition, f is differentiable on $(-1,1)$, and the derivative $f'(x) = \frac{1}{2\sqrt{x+1}}$ is unbounded on $[-1,1]$, so by the contrapositive of prop 26.3, f is not Lipschitzian on $(-1,1)$, so it is not on $[-1,1]$.

$f(x) = \sqrt[3]{x}$ is another example.

4) For each $x \in \mathbb{Z}$, consider the closed bounded interval $[x, x+1]$. Let D_x be the set of discontinuities on $[x, x+1]$. By th 27.2 (which is proved fully in the text as th 3.33, on pg. 117), each D_x is countable. Now, clearly $D = \bigcup_{x \in \mathbb{Z}} D_x$, so D is the countable union of countable sets. Thus D is itself countable.

5) Let $[c,d] \subseteq [a,b]$ be given. Assume that for every $x \in [c,d]$, f is discontinuous at x. $[c,d]$ is an uncountable set and $[c,d]$ is a subset of the set of discontinuities, so the set of discontinuities is uncountable. Contradiction. So there must be some $x \in [c,d]$ such that f is continuous at x.

6) Suppose that g is integrable. We use the definition of integrability to obtain a contradiction. Let $\varepsilon = \frac{1}{2}$. Then, there exists a $\delta > 0$ such that

$$|S(g, {}^tP_1) - S(g, {}^tP_2)| \leq |S(g, {}^tP_1) - \int_1^2 g| + |S(g, {}^tP_2) - \int_1^2 g| < \frac{1}{2} + \frac{1}{2} = 1 \quad (1)$$

whenever

$$\| {}^tP_1 \| < \delta \quad \text{and} \quad \| {}^tP_2 \| < \delta . \quad (2)$$

Take $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \delta$. Take $P \in \mathcal{P}([a,b])$ defined as:

$$P = \{ x_i : i = 0, \ldots, n_0 \}, \quad x_i = 1 + \frac{i}{n_0}, \quad i = 0, \ldots, n_0 .$$

Let ${}^tP_1 = \{(t_i, [x_{i-1}, x_i]) : i = 1, \ldots, n_0\}$ and ${}^tP_2 = \{(s_i, [x_{i-1}, x_i]) : i = 1, \ldots, n_0\}$

be two tagged partitions associated with P such that:

$$t_i \in \mathbb{Q} \quad , \quad s_i \notin \mathbb{Q} \quad \text{for } i = 1, \ldots, n_0 .$$

Then

$$S(g, {}^tP_1) = \sum_{i=1}^{n_0} g(t_i)(x_i - x_{i-1}) \geqslant \sum_{i=1}^{n_0} (x_i - x_{i-1}) = 2 - 1 = 1$$

as $g(t_i) \geqslant 1$ for $t_i \in [1, 2]$. Also,

$$S(g, {}^tP_2) = \sum_{i=1}^{n_0} g(s_i)(x_i - x_{i-1}) = 0 .$$

Hence,

$$|S(g, {}^tP_1) - S(g, {}^tP_2)| = 1 .$$

Contradiction with (1) .

7) Let f be defined on $[0,1]$ as follows:

$$f(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases}$$

Given $\varepsilon > 0$, there exists $\delta = \frac{\varepsilon}{2} > 0$ such that given ${}^tP$ where $\|{}^tP\| < \delta$, ${}^tP = \{(t_i, [x_{i-1}, x_i]) : i = 1, \ldots, n\}$ we have: if $x = 1$ is not a tag in ${}^tP$

$$|S(f, {}^tP) - 0| = |S(0, {}^tP) - 0| = 0 < \varepsilon .$$

if $x = 1$ is a tag in ${}^tP$

$$|S(f, {}^tP) - 0| = |(x_n - x_{n-1})| < \delta = \frac{\varepsilon}{2} < \varepsilon .$$

Thus f defined on $[0,1]$ is Riemann integrable, and

$$\int_0^1 f = 0 . \blacksquare$$