MTH 435 Homework 9 - Solutions

3) Suppose that f is not constant. Then there exist $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) \neq f(x_2)$. Without any loss of generality we may assume that $x_1 < x_2$. Since f is continuous in $[x_1, x_2]$ the Intermediate Value Theorem applies. Take v such that v is between $f(x_1)$ and $f(x_2)$ and $v \in \mathbb{R} \setminus \mathbb{Q}$. By the IVT, there exists $c \in [x_1, x_2]$ such that $f(c) = v$. Contradiction as f takes only rational values and v is irrational.

4) Consider the function $g(x) = f(x) - x$, $x \in [0,1]$.
If $f(0) = 0$, $x = 0$ is a fixed point. Suppose $f(0) \neq 0$. Then $f(0) > 0$ as f takes values in $[0,1]$. Thus, $g(0) > 0$. Since $f(x) \leq 1$ for all $x \in [0,1]$, in particular, $f(1) \leq 1$, we obtain $g(1) \leq 0$. $g(x)$ is continuous in $[0,1]$ as the difference of two continuous functions, and $g(0) > 0$, $g(1) \leq 0$. Thus

$$g(1) \leq 0 < g(0).$$

By the Intermediate Value Theorem, there exists $x_0 \in [0,1]$. Such that $g(x_0) = 0$. But then $f(x_0) = x_0$ and x_0 is a fixed point.

5) Suppose $L < M$. Take $\varepsilon = \dfrac{M-L}{2}$. Since $\lim_{n \to +\infty} a_n = L$, $\lim_{n \to +\infty} b_n = M$, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$|a_n - L| < \varepsilon \quad \text{for } n \geq N_1,$$

$$|b_n - M| < \varepsilon \quad \text{for } n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. The latter inequalities imply that for all $n \geq N$ we have

$$L - \varepsilon < a_n < L + \varepsilon \quad \text{and} \quad M - \varepsilon < b_n < M + \varepsilon.$$

But $L + \varepsilon = \dfrac{L+M}{2}$, $M - \varepsilon = \dfrac{L+M}{2}$. Hence, for $n \geq N$,

$$a_n < b_n.$$

Contradiction. Therefore, $L \geq M$.

Example: Let $a_n = \frac{1}{n}$, $b_n = 0$ for $n = 1, 2, \ldots$. We have $a_n > b_n$ for $n = 1, 2, \ldots$, yet $\lim\limits_{n \to +\infty} a_n = \lim\limits_{n \to +\infty} b_n = 0$.

6) Let $f(x) = x^2$, $I = (0, 1)$. f is continuous on $(0,1)$, yet f does not attain its minimum or maximum values in $(0,1)$. For every $x \in (0,1)$, there exist $y \in (0,1)$ such that $y > x$, thus $f(y) > f(x)$. Hence, the largest value is not attained. Similarly, the smallest value is not attained. Obviously, the effect is due to the fact that the interval $(0,1)$ is not closed.