MTH 435 Homework 8 - Solutions Fall 02

1) You prove Th 20.4 by minor modifications of the proof of Th. 19.1.

2) (a) Let $c \in \mathbb{R}$. Let f be defined in some interval (c, a). We say that

$$\lim_{x \to c+} f(x) = +\infty$$

if

$$\forall_{M>0} \; \exists_{\delta>0} \; \forall_{x \in (c,a)} \; (c < x < c+\delta \Rightarrow f(x) > M).$$

(b) Let $c \in \mathbb{R}$. Let f be defined in a deleted nbhd N_d of c. We say that

$$\lim_{x \to c} f(x) = -\infty$$

if

$$\forall_{M>0} \; \exists_{\delta>0} \; \forall_{x \in N_d} \; (0 < |x-c| < \delta \Rightarrow f(x) < -M).$$

(c) Let f be defined in some interval $(a, +\infty)$. Let $L \in \mathbb{R}$. We say that

$$\lim_{x \to +\infty} f(x) = L$$

if

$$\forall_{\varepsilon>0} \; \exists_{K \in \mathbb{R}} \; \forall_{x \in (a,+\infty)} \; (x > K \Rightarrow |f(x) - L| < \varepsilon).$$

(d) Let f be defined in some interval $(a, +\infty)$. We say

$$\lim_{x \to +\infty} f(x) = +\infty.$$

if

$$\forall_{M>0} \; \exists_{K \in \mathbb{R}} \; \forall_{x \in (a,+\infty)} \; (x > K \Rightarrow f(x) > M).$$

3) Let's first define limit at a point being $+\infty$.

Def: Let $c \in \mathbb{R}$, f be defined in a deleted nbhd N_d of c. We say that

$$\lim_{x \to c} f(x) = +\infty$$

if

$$\forall_{M>0} \ \exists_{\delta>0} \ \forall_{x \in N_d} \ (0<|x-c|<\delta \Rightarrow f(x) > M). \quad (1)$$

Consider $f(x) = \dfrac{1}{(x-1)^2}$, $c = 1$. We shall prove

$$\lim_{x \to 1} \frac{1}{(x-1)^2} = +\infty. \quad (2)$$

To prove (2), take an arbitrary $M>0$. Let $\delta = \sqrt{\frac{1}{M}}$. Let $x \in N_d$ be such that

$$0 < |x-1| < \delta.$$

Then

$$\frac{1}{(x-1)^2} = \frac{1}{|x-1|^2} > \frac{1}{\delta^2} = \frac{1}{\frac{1}{M}} = M.$$

Since M was arbitrary, (1) holds and (2) is proved.

4) Let $x_0 \in (0,1)$ be arbitrary. We shall prove $f(x_0)=0$. From the density of rationals, for every $n=1,2,\dots$ we can choose $q_n \in (0,1) \cap \mathbb{Q}$ such that

$$q_n \in (x_0 - \tfrac{1}{n}, x_0 + \tfrac{1}{n}). \quad (3)$$

(3) implies that $\lim_{n \to +\infty} q_n = x_0$. Since $q_n \in \mathbb{Q}$ for $n=1,2,\dots$ we have

$$f(q_n) = 0 \text{ for } n=1,2,\dots. \quad (4)$$

From Th. 19.1, $\lim f(q_n) = f(x_0)$. But $\lim_{n \to +\infty} f(q_n) = 0$ by (4).

Therefore, $f(x_0) = 0$.

5) By Prop. 21.2 and continuity of f and g in $\mathbb{R}$ we have

$$\lim_{x \to a} f(x) = f(a) \quad , \quad \lim_{x \to a} g(x) = g(a). \quad (5)$$

Take $\varepsilon = \dfrac{f(a) - g(a)}{2}$. Since $f(a) > g(a)$, $\varepsilon > 0$. By (5) there exist δ_1, δ_2 both positive and such that

$$|f(x) - f(a)| < \varepsilon \text{ whenever } |x - a| < \delta_1 , \quad (6)$$

$$|g(x) - g(a)| < \varepsilon \text{ whenever } |x - a| < \delta_2 . \quad (7)$$

Take $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$ and by (6) and (7)

$$|f(x) - f(a)| < \varepsilon \text{ and } |g(x) - g(a)| < \varepsilon \text{ whenever } |x - a| < \delta.$$

In other words, for all $x \in (a - \delta, a + \delta)$ we have

$$f(a) - \varepsilon < f(x) < f(a) + \varepsilon \quad , \quad g(a) - \varepsilon < g(x) < g(a) + \varepsilon . \quad (8)$$

By the choice of ε we have $f(a) - \varepsilon = g(a) + \varepsilon$. Hence, by (8) we obtain

$$f(x) > g(x) \text{ for all } x \in (a - \delta, a + \delta).$$

6) Straightforward from the definition of the limit of a sequence.