MTH 435 Homework 7 - Solutions Fall 02

2) We shall prove that $L \leq M$. The proof of $L \geq m$ is similar.
Suppose $L > M$. Let $\varepsilon = L - M$. Then $\varepsilon > 0$ and, since $\lim_{x \to c} f(x) = L$, there exists a $\delta > 0$ such that for all $x \in I$:

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad |x - c| < \delta, \ x \neq c. \quad (1)$$

Take an x such that $0 < |x - c| < \delta$, $x \in I$. Then from (1)

$$L - \varepsilon < f(x) < L + \varepsilon,$$

which implies $f(x) > L - (L - M)$. We obtain

$$f(x) > M.$$

Contradiction as $f(x) \leq M$ for all $x \in I \setminus \{c\}$.

3) We have to show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left|(x-1)^2 \cos \frac{1}{x-1}\right| < \varepsilon \quad \text{whenever } 0 < |x - 1| < \delta. \quad (2)$$

Take an $\varepsilon > 0$. Let $\delta = \min\{\varepsilon, 1\}$. Let x be such that

$$0 < |x - 1| < \delta. \quad (3)$$

Since $\delta \leq 1$, (3) implies that $|x-1|^2 \leq |x-1|$. Hence:

$$\left|(x-1)^2 \cos \frac{1}{x-1}\right| = |x-1|^2 \left|\cos \frac{1}{x-1}\right| \leq |x-1|^2 \leq |x-1| < \delta \leq \varepsilon.$$

The latter inequality follows as $\delta = \min\{\varepsilon, 1\}$. Since ε and x were arbitrary, (2) follows.

4) Take two sequences:

$$x_n = \frac{1}{\frac{\pi}{2} + n\pi}, \quad y_n = \frac{1}{2\pi n} \quad \text{for } n = 1, 2, \dots$$

We have

$$\lim_{n \to +\infty} x_n = \lim_{n \to +\infty} y_n = 0.$$

If $\lim_{x \to 0} \cos \frac{1}{x}$ existed, $\lim_{x \to 0} \cos \frac{1}{x} = L$, it would have to be

$$\lim_{n \to +\infty} \cos \frac{1}{x_n} = \lim_{n \to +\infty} \cos \frac{1}{y_n} = L. \quad (3)$$

Yet,

$$\cos\frac{1}{x_n} = \cos\left(\frac{\pi}{2}+n\pi\right)=0 \quad \text{for } n=1,2,\ldots,$$

$$\cos\frac{1}{y_n} = \cos(2\pi n)=1 \quad \text{for } n=1,2,\ldots .$$

Hence, (3) cannot be satisfied. We conclude that

$$\lim_{x\to 0} \cos\frac{1}{x}$$

does not exist.

6) (a) $\lim_{x\to 0} f(x)=0$. (4)

Indeed, let $\varepsilon>0$ be given. Take $\delta=\varepsilon$. Take x such that

$$0<|x|<\delta. \qquad (5)$$

If x is rational, $f(x)=x$. If x is irrational, $f(x)=0$.

In either case:

$$|f(x)|<\varepsilon.$$

Since ε and x were arbitrary, (4) is proved.

(b) For $c\neq 0$ $\lim_{x\to c} f(x)$ does not exist. Indeed, take a sequence $\{q_n\}_{n=1}^{\infty}$ such that $q_n\to c$ as $n\to+\infty$, $q_n\in\mathbb{Q}$. Such sequence exists from the density of rationals. From the density of irrationals, there exists a sequence $\{p_n\}_{n=1}^{\infty}$ such that $p_n\to c$ as $n\to+\infty$, $p_n\in\mathbb{R}\setminus\mathbb{Q}$. From the definition of f:

$$f(q_n)=q_n, \quad f(p_n)=0 \quad \text{for } n=1,2,\ldots$$

Thus

$$\lim_{n\to+\infty} f(q_n)=c,\ c\neq 0, \quad \lim_{n\to+\infty} f(p_n)=0$$

Therefore, $\lim_{x\to c} f(x)$ does not exist.