Homework 5 - Solutions

1) The inequality
$$|x|-|y| \leq |x-y| \quad (1)$$
is equivalent to
$$|x| \leq |x-y|+|y|. \quad (2)$$
Use the triangle inequality $|a+b| \leq |a|+|b|$ with $a=x-y$, $b=y$ (2) follows. Thus (1).

2) Done in class.

3) Done in class.

4) The sequence is convergent. Namely:
$$\lim_{n\to+\infty} \frac{(-1)^n}{n} = 0. \quad (3)$$
Indeed, take $\varepsilon>0$. Let $N > \frac{1}{\varepsilon}$. Then for $n \geq N$ we have
$$\left|\frac{(-1)^n}{n}\right| = \frac{1}{n} < \varepsilon.$$
Thus (3).

5) Take $\varepsilon>0$. Let $N > \frac{1}{\varepsilon}$. Then for $n \geq N$ we have
$$\left|\left(1-\frac{1}{n^2}\right)-1\right| = \left|-\frac{1}{n^2}\right| = \frac{1}{n^2} < \frac{1}{n} < \varepsilon.$$

6) $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ are bounded. Hence there exist $K, M \in \mathbb{R}$ such that
$$|x_n| \leq K \quad , \quad |y_n| \leq M \quad \text{for all } n \in \mathbb{N}.$$
Then for all $n \in \mathbb{N}$:
$$|x_n+y_n| \leq |x_n|+|y_n| \leq K+M \quad , \quad |x_n-y_n| \leq |x_n|+|y_n| \leq K+M$$
$$|x_n y_n| = |x_n||y_n| \leq K\cdot M.$$
Thus, all the required sequences are bounded.

7) Done in class.