Solutions to Homework 4, Problems 1–5.

1) Note we are not assuming that S is dense. We have to prove that (i) $\Leftrightarrow$ (ii).

"$\Rightarrow$" Assume (i). We have to prove (ii). Let's rewrite (i) in an equivalent form:

(i) $\forall_{\alpha \in \mathbb{R}} \ \forall_{\varepsilon > 0} \ \exists_{r \in S} \ r \in (\alpha - \varepsilon, \alpha + \varepsilon)$

so we won't confuse a's of (i) and (ii). We have to prove

(ii) $\forall_{\substack{a, b \in \mathbb{R} \\ a < b}} \ \exists_{r \in S} \ a < r < b.$

Take $a, b \in \mathbb{R}$ such that $a < b$. Let $m = \frac{a+b}{2}$ be the midpoint between a and b, let $\varepsilon = \frac{b-a}{2}$. Then $a = m - \varepsilon$, $b = m + \varepsilon$. By (i), with $\alpha = m$, we have that there exists $r \in S$ such that $r \in (m - \varepsilon, m + \varepsilon)$. Thus, $a < r < b$ and (ii) holds.

"$\Leftarrow$" Assume (ii). We have to prove (i). Let $\alpha \in \mathbb{R}$ and $\varepsilon > 0$ be given. Take $a = \alpha - \varepsilon$, $b = \alpha + \varepsilon$. Then $a < b$ and, by (ii), there exists $r \in S$ such that $a < r < b$. In other words, $\alpha - \varepsilon < r < \alpha + \varepsilon$; that is, $r \in (\alpha - \varepsilon, \alpha + \varepsilon)$. Hence, (i) holds.

2) We have to prove that S satisfies (i) of Pr. 1 or, equivalently, (ii). Take $a, b \in \mathbb{R}$, $a < b$. Then

$$\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}.$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $q = \frac{m}{n}$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$ such that

$$\frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}}.$$

Thus:

$$a < \frac{\sqrt{2}\,m}{n} < b.$$

Therefore, S is dense in $\mathbb{R}$.

3) $\mathbb{Z}$ is not dense in $\mathbb{R}$. By Prop 11.1, for any $m \in \mathbb{Z}$, there are no integers in $(m, m+1)$. Thus, $\mathbb{Z}$ does not satisfy (ii) of Pr. 1.

4) Let $B = \mathbb{Q} \cap (0,1)$. Since $B \subseteq (0,1)$, 1 is an upper bound for B, 0 a lower bound.

We prove first that $1 = \sup B$. Indeed, suppose there is an upper bound p for B such that $p < 1$. Of course, $p > 0$. By density of rationals there exists $q \in \mathbb{Q}$ such that $p < q < 1$. But then $q \in B$ and p is not an upper bound for B. Thus, $1 = \sup B$.

We prove that $0 = \inf B$. Suppose there is a lower bound r for B such that $r > 0$. Of course, $r < 1$. By density of rationals, there exists $q \in \mathbb{Q}$ such that

$$0 < q < r.$$

But then $q \in B$ and r is not a lower bound for B. Thus $0 = \inf B$.

5) For every $t \in T$ we have $t \le s_1$. For every $v \in V$ we have $v \le s_2$. Thus for every $t+v \in S$ we have $t+v \le s_1+s_2$. Hence, s_1+s_2 is an upper bound for S. We shall use Th 11.2 to show that

$$s_1+s_2 = \sup S. \qquad (1)$$

We have shown (a) of Th 11.2, now we'll show (b). Let $\varepsilon > 0$ be given. Applying Th 11.2 to T and V, we obtain that there exist $t_0 \in T$ and $v_0 \in V$ such that

$$s_1 - \frac{\varepsilon}{2} < t_0 \le s_1 \quad , \quad s_2 - \frac{\varepsilon}{2} < v_0 \le s_2 . \quad (2)$$

Indeed $s_1 = \sup T$, $s_2 = \sup V$, hence they both satisfy (b) of Th 11.2. Adding the inequalities (2), we obtain

$$(s_1+s_2) - \varepsilon < t_0+v_0 \le s_1+s_2 .$$

Thus, s_1+s_2 satisfies (b) of Th 11.2, and (1) holds.