Solutions to Homework 3, Problems 5–7

5) Since $A \neq \emptyset$, there exists $a \in A$. Hence:

$$\inf A \leq a \leq \sup A.$$

Thus:

$$\inf A \leq \sup A.$$

Let $A = \{a\}$ be a set with one element. From the definition of the supremum and the infimum, we obtain easily:

$$\inf A = a = \sup A.$$

6) (a) We have to prove for a fixed $a \in \mathbb{R}$, $a \neq -1$:

$$1 + a + a^2 + \ldots + a^n = \frac{1-a^{n+1}}{1-a} \quad \text{for all } n \in \mathbb{N}. \quad (1)$$

We use FIP.

(i) For $n = 1$ the formula becomes:

$$1 + a = \frac{1-a^2}{1-a}.$$

Since $1 - a^2 = (1-a)(1+a)$, the formula is true.

(ii) Let $k \in \mathbb{N}$ be given. Assume:

$$\text{(IA)} \quad 1 + a + \ldots + a^k = \frac{1-a^{k+1}}{1-a}.$$

We have to prove that

$$1 + a + \ldots + a^k + a^{k+1} = \frac{1-a^{(k+1)+1}}{1-a}. \quad (2).$$

Indeed, by (IA):

$$1 + a + \ldots + a^k + a^{k+1} = (1 + a + \ldots + a^k) + a^{k+1} =$$

$$= \frac{1-a^{k+1}}{1-a} + a^{k+1} = \frac{1-a^{k+1}+a^{k+1}-a^{k+2}}{1-a} = \frac{1-a^{k+2}}{1-a}.$$

Thus (2) holds, and, by FIP, (1) holds.

(6) Let $a \in \mathbb{R}$, $a > -1$. We have to prove that

$$(1+a)^n \geqslant 1+na \quad \text{for all } n \in \mathbb{N}. \quad (3)$$

We use FIP.

(i) For $n=1$ the formula (3) becomes

$$1+a \geqslant 1+a$$

and is true.

(ii) Let $k \in \mathbb{N}$ be given. Assume that

$$(1+a)^k \geqslant 1+ka \quad (4).$$

We have to prove that

$$(1+a)^{k+1} \geqslant 1+(k+1)a. \quad (5)$$

Indeed, by (IA):

$$(1+a)^{k+1} = (1+a)^k(1+a) \geqslant (1+ka)(1+a) =$$

$$= 1+ka+a+ka^2 = 1+(k+1)a+ka^2 \geqslant$$

$$\geqslant 1+(k+1)a.$$

The latter inequality holds as $ka^2 \geqslant 0$. Thus (5) holds, and, by FIP, (3) holds.

7) For every $ka \in B$ we have $ka \leqslant ks$ as $k>0$ and $a \leqslant s$ for all $a \in A$. Thus ks is an upper bound for B. Suppose $p < ks$ is an upper bound for B. Then, $ka \leqslant p < ks$ for all $a \in A$. Thus, for all $a \in A$:

$$a \leqslant \frac{1}{k}p < s.$$

Contradiction as $s = \sup A$. If $k \leqslant 0$, $ks = \inf B$. Prove it yourself.