March 31
April 2

16.4 Double Integrals in Polar Coordinates

We are back to double integrals:

$$\int_R f(x,y)\,dA,$$

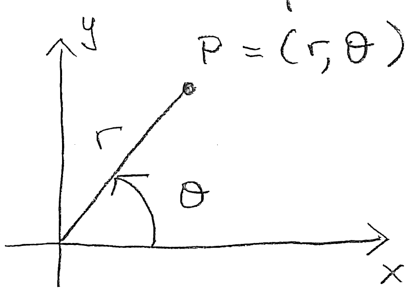
where $f(x,y)$ is a given function, R a given region on the xy-plane. As you saw, the key to setting up iterated integrals is the shape of R. So far we were using the rectangular coordinates to describe regions of integration. Many regions have a much easier description in polar coordinates. A quick refresher.

Polar Coordinates:

In polar coordinates every point P on the xy-plane is described by two coordinates:

$$P = (r, \theta)$$

where r is the distance of P from the origin O (called the pole), θ is the angle, in radians, between OP and the positive x-axis (called the polar axis) measured ccw:



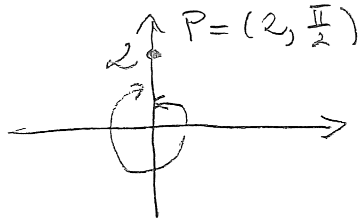
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}, \quad x^2+y^2 = r^2$$

Ex: Give polar coordinates of $P=(0,2)$ in rectangular coords.

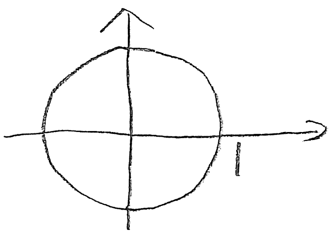
$$P = (2, \tfrac{\pi}{2}) \text{ also } P = (2, -\tfrac{3\pi}{2}),\ P = (2, \tfrac{5\pi}{2})\ldots$$

The second polar coordinate is not unique:

If $P = (r, \theta)$, then $P = (r, \theta + 2\pi n)$, $n = 0, \pm 1, \pm 2, \ldots$

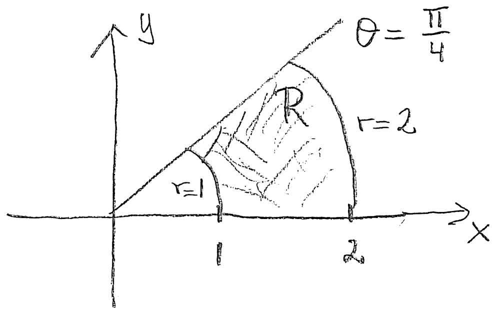


Ex: Find the equation of the unit circle in polar coordinates.



In rectangular:	In polar:
$x^2 + y^2 = 1$	$r = 1$

Ex: Describe the region R: $1 \leq r \leq 2$, $0 \leq \theta \leq \frac{\pi}{4}$.
(A "rectangle" in polar coordinates.).



R has a much more complicated decription in rectangular coordinates.

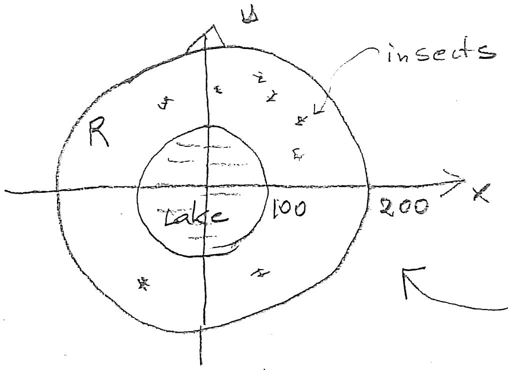
Ex: The density of insects in a circular region

$$100 \leq \sqrt{x^2+y^2} \leq 200$$

around a circular lake $0 \leq \sqrt{x^2+y^2} \leq 100$, x, y in meters, is given by

$$d(x,y) = \frac{1000}{\sqrt{x^2+y^2}} \quad \frac{\text{insects}}{m^2}$$

Find the total number of insects in the region.



$$\text{Total insects} = \int_R d(x,y)\,dA.$$

The density, d, depends only on the distance from the lake.

R is not easy to decribe in rectanglar coordinates.

In polar, though:

$$R: \quad 100 \leq r \leq 200, \quad 0 \leq \theta \leq 2\pi$$

How to rewrite the double integral in polar coordinates?

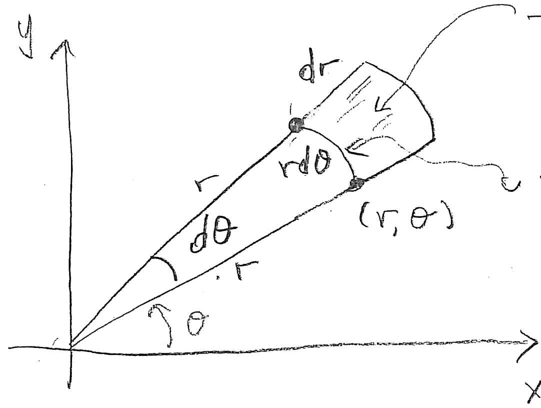
$$d(x,y) = \frac{1000}{\sqrt{x^2+y^2}} = \frac{1000}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}} = \frac{1000}{r}$$

In other words:

$$\int_R d(x,y)\,dA = \int_R d(r\cos\theta, r\sin\theta)\,dA$$

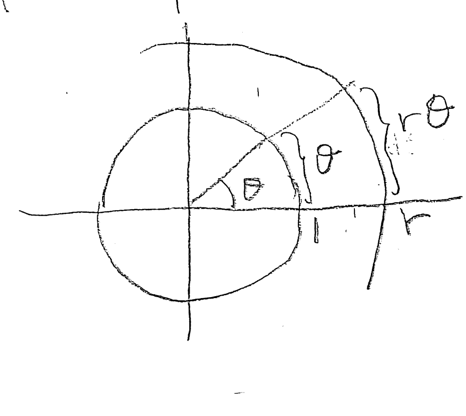
What is 'dA' in polar coordinates - the element of area?

'dA' is the infinitesimal change in area corresponding to infinitesimal changes dr, $d\theta$ in r and θ.



The change in area, dA, corresponding to the changes dr, $d\theta$.

$r\,d\theta$ by the definition of radian measure.

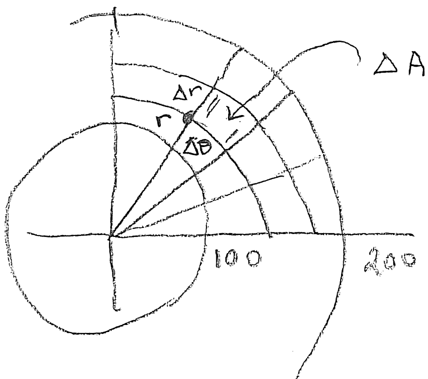


$$\underline{dA = r\,d\theta\,dr}$$

In polar coordinates the element of area depends on r:

$$dA = r\,d\theta\,dr = r\,dr\,d\theta$$

If you were calculating the integral over the region in polar coordinates, you would subdivide the region into small polar "rectangles":



$\Delta A \cong r \cdot \Delta\theta \cdot \Delta r$

Insects in the little "rectangle" $\cong d(r,\theta) \cdot r \cdot \Delta\theta \cdot \Delta r$

$$\text{Total insects} = \lim_{\substack{\Delta\theta \to 0 \\ \Delta r \to 0}} \sum d(r,\theta) \cdot r\,\Delta\theta \cdot \Delta r$$

Thus:

$$\text{Total insects} = \int_R d\,dA = \int_R d(x,y)\,dx\,dy = \int_R d(r\cos\theta, r\sin\theta) \cdot r\,dr\,d\theta =$$

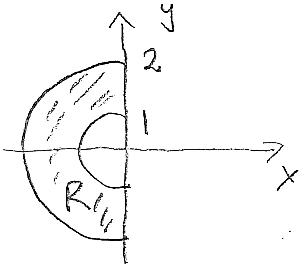
$$= \int_0^{2\pi} \int_{100}^{200} \frac{1000}{r} \cdot r\,dr\,d\theta = \int_0^{2\pi} 100{,}000\,d\theta = 2\pi \cdot 100{,}000 =$$

$\cong 628{,}318$ insects.

In general:

$$\int_R f(x,y)\,dx\,dy = \int_R f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Ex: Write the integral $\int_R f \, dA$ as an iterated integral in polar coordinates, where R is

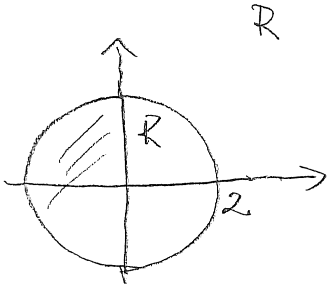


$$\int_R f \, dA = \int_1^2 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} f(r\cos\theta, r\sin\theta)\, r \, d\theta \, dr =$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_1^2 f(r\cos\theta, r\sin\theta)\, r \, dr \, d\theta .$$

Whichever is easier

Ex: $\int_R \sin(x^2+y^2)\, dA$, $R: x^2+y^2 \le 4$



Convert to polar:

$R: 0 \le r \le 2, \quad 0 \le \theta \le 2\pi$

$\sin(x^2+y^2) = \sin(r^2)$.

$$\int_R \sin(x^2+y^2)\, dA = \int_0^2 \int_0^{2\pi} \sin(r^2) . r . d\theta \, dr =$$

$$= \int_0^2 2\pi r \sin(r^2)\, dr = -\pi \cos(r^2) \Big|_0^2 =$$

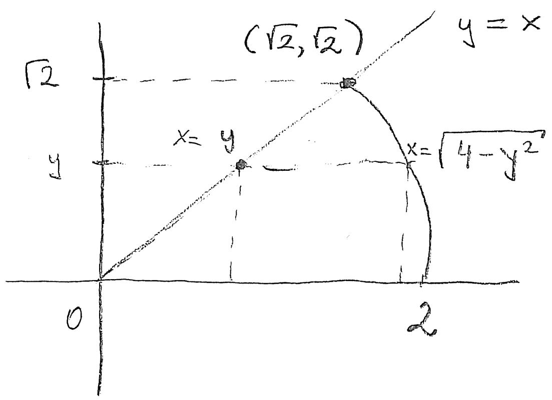
$$= \underline{-\pi \cos(4) + \pi}$$

Ex: Convert to polar coordinates. Evaluate.

$$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} (xy)\,dx\,dy$$

- Sketch the region R
- Describe the region in polar coordinates
- Set up an iterated integral in polar coordinates.

R: $0 \le y \le \sqrt{2}$, $y \le x \le \sqrt{4-y^2}$



$x = \sqrt{4-y^2}$

$x^2 + y^2 = 4$

Circle with radius 2.

$(\sqrt{2}, \sqrt{2})$ is on the circle.

So

R : $0 \le r \le 2$, $0 \le \theta \le \frac{\pi}{4}$

$$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} (xy)\,dx\,dy = \int_0^{\frac{\pi}{4}} \underbrace{\int_0^{2} r^2 \cos\theta \sin\theta \cdot r\,dr}\,d\theta =$$

$$= \int_0^{\frac{\pi}{4}} 2\sin(2\theta)\,d\theta = -\cos(2\theta)\Big|_0^{\frac{\pi}{4}} = 0 - (-1) = \underline{1}$$

$$\int_0^2 r^3 \frac{1}{2}\sin(2\theta)\,dr = \frac{1}{2}\sin(2\theta)\left(\frac{1}{4}r^4\Big|_0^2\right) = \frac{1}{2}\sin(2\theta)\cdot 4 =$$

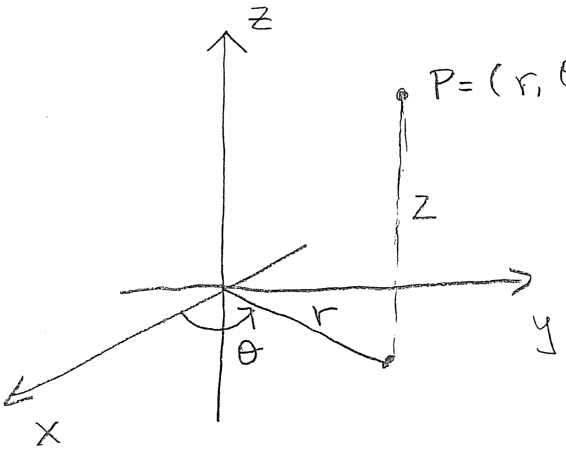
$$= 2\sin(2\theta)$$

16.5 Triple Integrals in Spherical and Cylindrical Coords

Many solids in the xyz-space have much easier decription in cylindrical or spherical coordinates than in rectangular coordinates.

Cylindrical coordinates

Very simple: polar coordinates on the xy-plane, z stays the same.



$0 \leq r \leq +\infty$

$0 \leq \theta \leq 2\pi$

$-\infty < z \leq +\infty$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} , \quad x^2+y^2=r^2$$

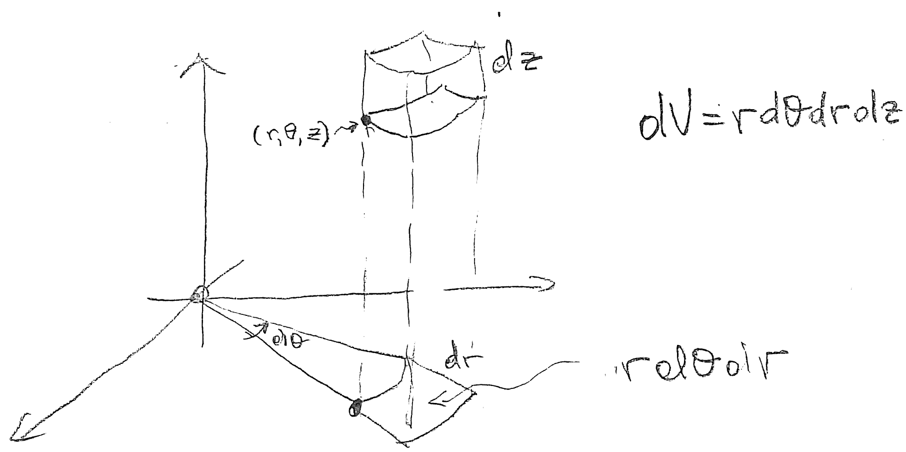
How do we convert a triple integral from rectangular to cylindrical coordinates? Easy.

$$\int_W f(x,y,z)\,dV = \int_W f(r\cos\theta, r\sin\theta, z)\, r\, \underbrace{d\theta\, dr\, dz}_{\text{or any other order.}}$$

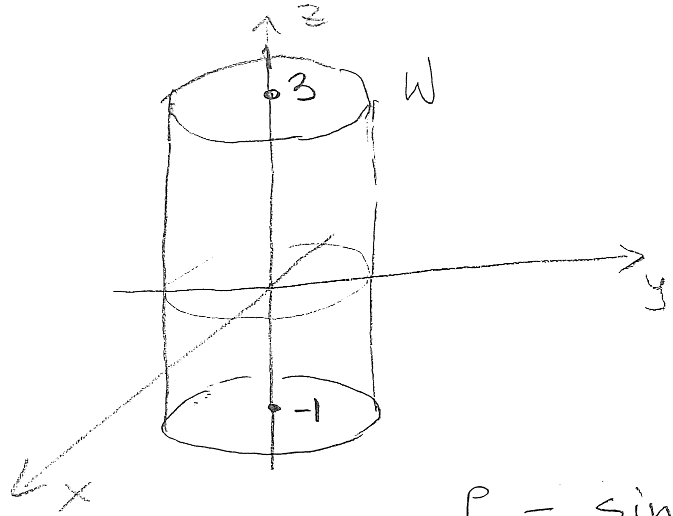
In other words, the element of Volume, dV, in cylindrical coordinates is:

$$\underline{dV = r\,d\theta\,dr\,dz}$$ (or any permutation of $d\theta\,dz\,dr$)

Why? Let's look at the infinitesimal change in V corresponding to infinitesimal changes: $d\theta$, dr, dz:



Ex 1. Calculate $\int_W f(x,y,z)\,dV$ where $f(x,y,z) = \sin(x^2+y^2)$, W is the solid cylinder with height 4 and the base of radius 1 centered on the z-axis, on the plane $z=-1$.



In cylindrical coordinates:

$W: \ 0 \le r \le 1, \ 0 \le \theta \le 2\pi,$

$-1 \le z \le 3$

$f = \sin(r^2)$

The integral:

$$\int_0^1 \int_{-1}^3 \int_0^{2\pi} \sin(r^2)\, r \, d\theta \, dz \, dr =$$

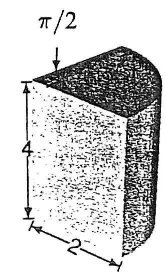
$$= \int_0^1 \int_{-1}^3 r\sin(r^2) \cdot 2\pi \, dz \, dr = \int_0^1 8\pi r \sin(r^2) dr =$$

$$= -4\pi \cos(r^2) \Big|_0^1 = \underline{-4\pi\cos(1) + 4\pi} \approx 5.78$$

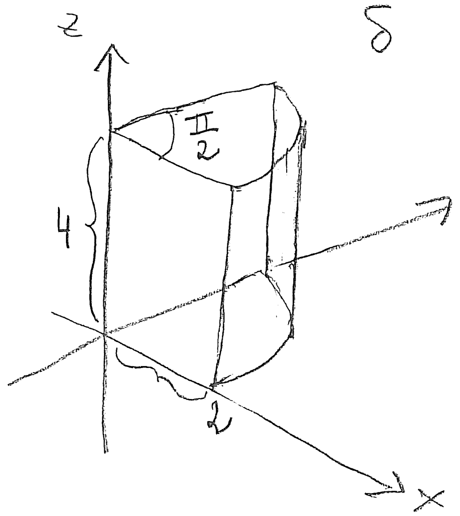
Ex 2:

For Problems 5–9, choose a set of coordinate axes, and then set up the three-variable integral in an appropriate coordinate system for integrating a density function δ over the given region.

6.



Let's choose an xyz-coordinate system so that the solid fits into the first octant. Then let's take the cylindrical coordinate system corresponding to this xyz-system as our solid is a wedge of a cylinder.



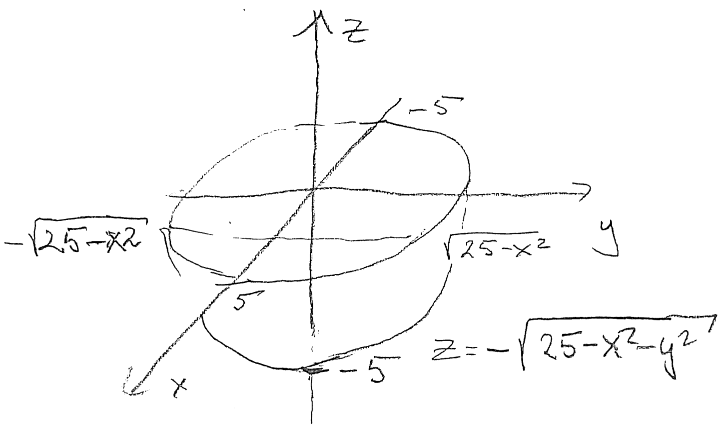
δ - the density function would have to be expressed in this coordinate system.

$$\int_W \delta(x,y,z)\,dV = \int_0^4 \int_0^{\frac{\pi}{2}} \int_0^2 \delta(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz$$

Cylindrical solids have easy decriptions in the cylindrical coords. How about spherical solids?

Ex: Describe the solid of integration for the triple integral:

$$\int_{-5}^{5} \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \int_{-\sqrt{25-x^2-y^2}}^{0} f(x,y,z)\, dz\, dy\, dx$$



The bottom half of the sphere

$x^2+y^2+z^2=25$

It depends what $f(x,y,z)$ is but, in general, this integral is very hard to find. We need a coordinate system in which spheres have an easy description.

Spherical Coordinates

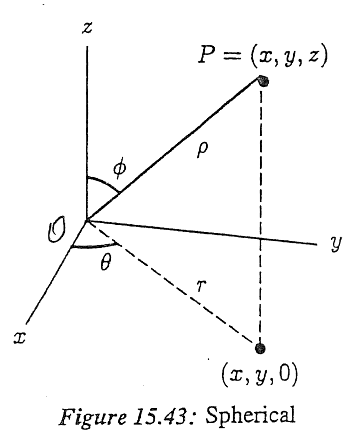


Figure 15.43: Spherical coordinates

Relation between Cartesian and Spherical Coordinates

Each point in 3-space is represented using $0 \le \rho < \infty, 0 \le \phi \le \pi$, and $0 \le \theta \le 2\pi$.

$(r = \rho \sin \Phi)$

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi.$$

Also, $\rho^2 = x^2+y^2+z^2$.

$P = (\rho, \theta, \Phi)$, $\rho = \text{dist}(P, O)$, θ the same as in cylindrical coordinates, Φ the angle between the positive z-axis and OP.

dV in spherical coordinates:

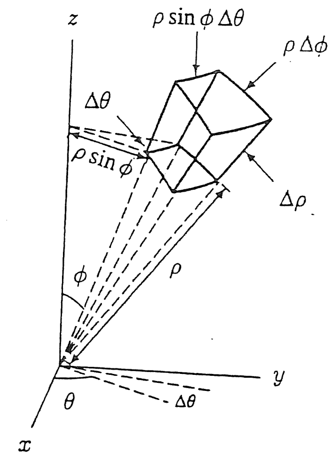


Figure 15.47: Volume element in spherical coordinates

When computing integrals in spherical coordinates, put $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$. Other orders of integration are also possible.

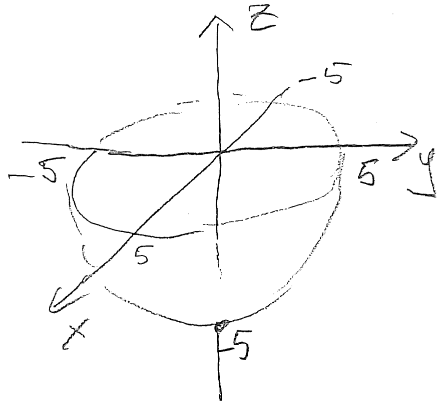
$$dV = \rho^2 \sin(\Phi)\, d\rho\, d\Phi\, d\theta$$

Ex: Evaluate

$$\int_W f(x,y,z)\, dV$$

where $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$, W is the bottom half of the sphere of radius 5 centered at the origin.

The same solid we just looked at. The integral practically impossible to do in rectangular coordinates.



$W: 0 \le \rho \le 5, \quad 0 \le \theta \le 2\pi$

$\frac{\pi}{2} \le \Phi \le \pi$

$$f(\rho, \theta, \Phi) = \frac{1}{\rho}$$

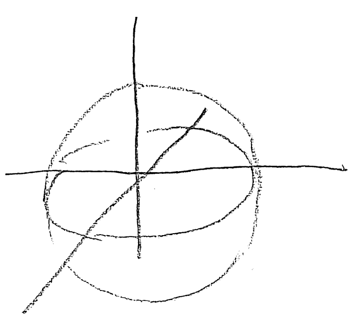
$$\int_W f\, dV = \int_0^5 \int_0^{2\pi} \int_{\frac{\pi}{2}}^{\pi} \frac{1}{\rho} \cdot \rho^2 \sin(\phi)\, d\phi\, d\theta\, d\rho =$$

$$= \int_0^5 \int_0^{2\pi} \int_{\frac{\pi}{2}}^{\pi} \rho \sin\Phi\, d\Phi\, d\theta\, d\rho =$$

$$= \int_0^5 \int_0^{2\pi} \underbrace{\left(-\rho\cos\Phi \Big|_{\frac{\pi}{2}}^{\pi}\right)}_{= \rho} d\theta\, d\rho =$$

$$= \int_0^5 \int_0^{2\pi} \rho\, d\theta\, d\rho = \int_0^5 2\pi\rho\, d\rho = \pi\rho^2 \Big|_0^5 = \underline{25\pi}$$

Ex: Find the volume of the sphere of radius R.



$S: 0 \le \rho \le R, \quad 0 \le \theta \le 2\pi, \quad 0 \le \Phi \le \pi$

$$V = \int_0^R \int_0^{\pi} \int_0^{2\pi} \rho^2 \sin\Phi\, d\theta\, d\Phi\, d\rho =$$

$$= \int_0^R \int_0^\pi 2\pi \rho^2 \sin\Phi \, d\Phi \, d\rho =$$

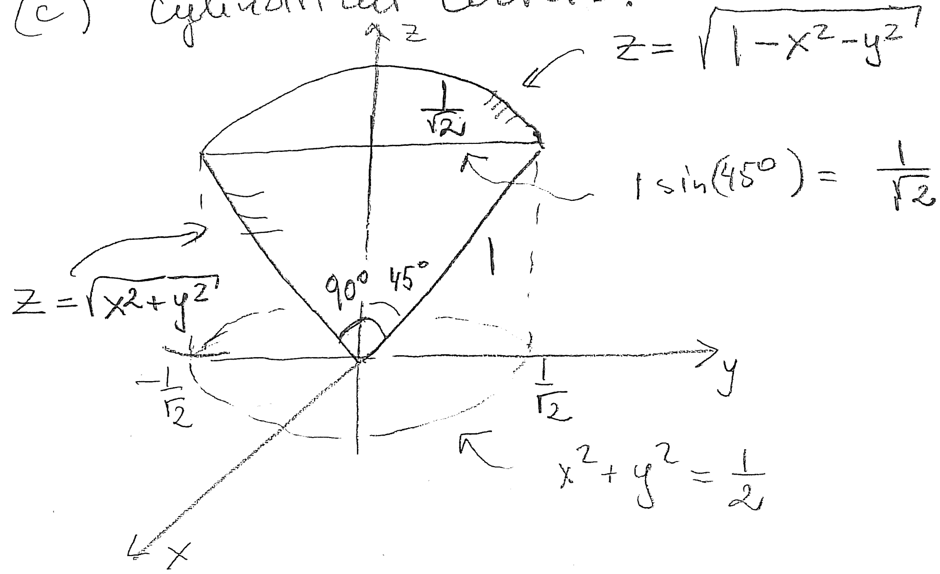
$$= \int_0^R 2\pi\rho^2 \Big(\underbrace{-\cos\Phi \Big|_0^\pi}_{2} \Big) d\rho = \int_0^R 4\pi \rho^2 d\rho =$$

$$= 4\pi \cdot \frac{1}{3} \rho^3 \Big|_0^R = 4\pi \cdot \frac{1}{3} R^3 = \underline{\underline{\frac{4}{3}\pi R^3}}$$

Ex: Let W be a cone with the angle $90°$ at the vertex topped by a sphere of radius 1. Write the iterated integral for

$$\int_W 1 \, dV$$

in (a) rectangular coords (b) spherical coords (c) cylindrical coords.



(a)

$$\int_W 1\, dV = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} 1 \, dz\, dy\, dx$$

(b)

$$\int_W 1 dV = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{1} 1\, \rho^2 \sin\Phi \, d\rho\, d\Phi\, d\theta$$

(c)

$$\int_W 1 dV = \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} \int_r^{\sqrt{1-r^2}} 1\, r\, dz\, dr\, d\theta$$