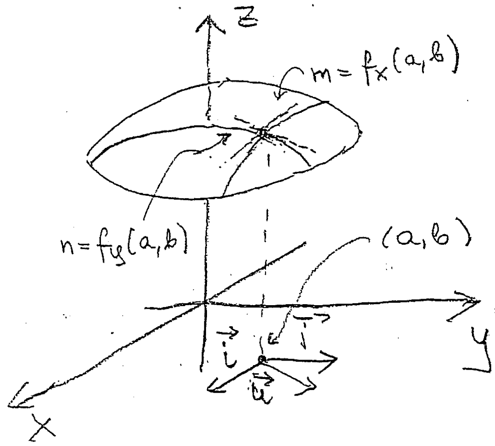
14.4 Gradients. Directional Derivatives

Let $z = f(x,y)$ be given. The partial derivatives $f_x(a,b)$, $f_y(a,b)$ at some point (a,b) give the rates of change in the direction of x and the direction of y:



$f_x(a,b)$ gives us the rate of change when we walk from (a,b) in the direction of $\vec{i}$, $f_y(a,b)$ if we walk in the direction of $\vec{j}$.

What is the rate of change if we move from (a,b) in the direction of some other unit vector $\vec{u}$?

This rate of change is called the directional derivative of $f(x,y)$ at (a,b) in the direction of $\vec{u}$.

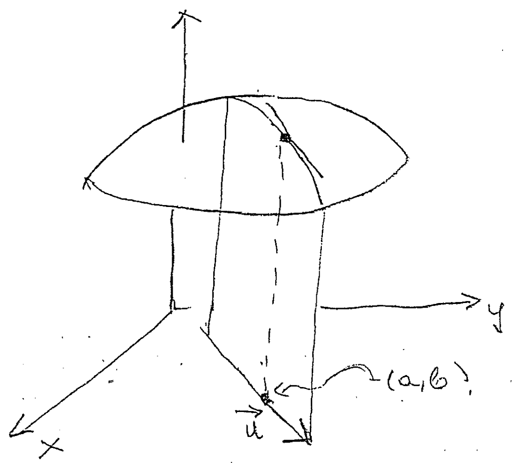
<u>Def</u>: Let $\vec{u} = u_1\vec{i} + u_2\vec{j}$, $\|\vec{u}\| = 1$, $z = f(x,y)$, (a,b) be given. The directional derivative of $f(x,y)$ at (a,b) in the direction of $\vec{u}$ is:

$$f_{\vec{u}}(a,b) = \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

$$(a,b) \longrightarrow (a+hu_1, b+hu_2).$$

Note: The definition talks about both-sided limit "$\lim_{h\to 0}$". Thus h can be positive or negative.

The limit exists if the cross-section of the graph by the vertical plane containing $\vec{u}$ and $(a, b, f(a,b))$ has the tangent line at $(a, b, f(a,b))$:



Thus $f_{\vec{u}}(a,b)$ also tells us that if we go in the opposite direction, $-\vec{u}$, $f(x,y)$ will be changing at the rate $-f_{\vec{u}}(a,b) = f_{-\vec{u}}(a,b)$

Indeed:

$$f_{-\vec{u}}(a,b) = \lim_{h\to 0} \frac{f(a+h(-u_1), b+h(-u_2)) - f(a,b)}{h} =$$

$$= \lim_{h\to 0} \frac{f(a+(-h)u_1, b+(-h)u_2) - f(a,b)}{-h} =$$

$$= -f_{\vec{u}}(a,b).$$

Note, the displacement vector from (a, b) to $(a+hu_1, b+hu_2)$ is $\vec{d} = (h\,u_1)\vec{i} + (h\,u_2)\vec{j} = h \cdot \vec{u}$. So $\|\vec{d}\| = h$ (or more precisely $|h|$).

This is why we need $\vec{u}$ to be the unit vector so the magnitude of h is the magnitude of the displacement. Geometrically, $f_{\vec{u}}(a, b)$ is the slope of the cross-section with the vertical plane parallel to $\vec{u}$ through $(a, b, f(a, b))$. Clearly:

$$f_x(a, b) = f_{\vec{i}}(a, b), \quad f_y(a, b) = f_{\vec{j}}(a, b).$$

<u>Def</u>: Let $f(x, y)$, (a, b), and a vector $\vec{v}$ be given. Then

$$f_{\vec{v}}(a, b) = f_{\frac{\vec{v}}{\|\vec{v}\|}}(a, b).$$

How do we calculate directional derivatives? Using the so-called gradient vector.

<u>Def</u>: The gradient vector of $f(x, y)$ at (a, b) is defined as

$$\operatorname{grad} f(a, b) = f_x(a, b)\,\vec{i} + f_y(a, b)\,\vec{j}.$$

In general:

$$\text{grad} f(x,y) = f_x(x,y)\vec{i} + f_y(x,y)\vec{j}$$

or

$$\text{grad} f = f_x\vec{i} + f_y\vec{j}$$

Ex: Let $f(x,y) = x^3y + 3y + x$. Find grad $f(x,y)$. Find grad $f(1,0)$.

$$f_x = 3x^2y + 1, \quad f_y = x^3 + 3$$

$$\text{grad} f = (3x^2y+1)\vec{i} + (x^3+3)\vec{j}$$

$$\text{grad} f(1,0) = \vec{i} + 4\vec{j}$$

grad f is a vector!

Th: Let $f(x,y)$ be differentiable at (a,b), $\vec{u}$ be the unit vector. Then

$$f_{\vec{u}}(a,b) = \text{grad} f(a,b)\cdot\vec{u} =$$

$$= f_x(a,b)\,u_1 + f_y(a,b)\,u_2.$$

Gradient is also denoted:

$$\text{grad} f = \nabla f$$

Ex: Let $\vec{u} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$, $f(x,y) = x^2+y^2$.

Find $f_{\vec{u}}(1,0)$.

We use gradient.

$$\text{grad } f(x,y) = 2x\vec{i} + 2y\vec{j}$$

$$\text{grad } f(1,0) = 2\vec{i}$$

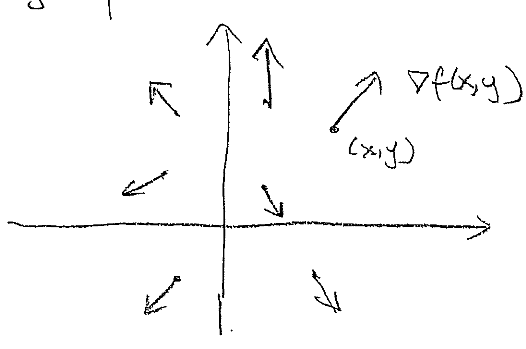
$$f_{\vec{u}}(1,0) = 2\vec{i}\cdot\vec{u} = 2\cdot\frac{1}{\sqrt{2}} = \sqrt{2}.$$

So finding directional derivatives is as easy as finding gradients.

Let $f(x,y)$, differentiable, be given. Then we have the gradient vector at each point:

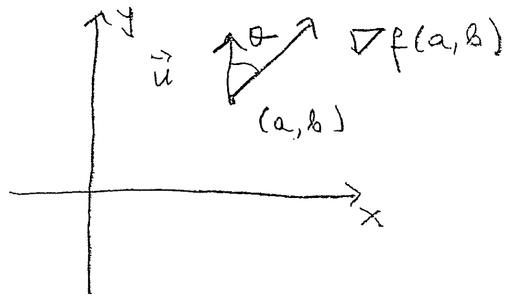
$$\nabla f(x,y)$$

So we have a vector field – the gradient field on the xy-plane:



Gradient Geometrically

Let's have $f(x,y)$, (a,b), $\nabla f(a,b)$:



How about directional derivatives at (a,b)?

Let $\vec{u}$ be a unit vector.

$$f_{\vec{u}}(a,b) = \nabla f(a,b) \cdot \vec{u} = \|\nabla f(a,b)\| \cdot \underbrace{\|\vec{u}\|}_{=1} \cos(\theta)$$

$f_{\vec{u}}(a,b)$ is the largest if $\cos(\theta) = 1$; that is, when $\theta = 0$, so $\vec{u}$ is parallel to $\nabla f(a,b)$.

Thus:

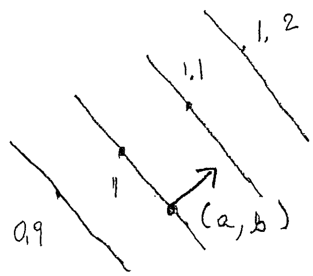
<u>Th</u>: Let $f(x,y)$ be differentiable at (a,b), $\nabla f(a,b) \neq \vec{0}$.
Then:

(1) $\nabla f(x,y)$ points in the direction of the largest rate of change of f at (a,b). (The direction of the fastest growth.)

(2) This largest rate of change is $\|\nabla f(a,b)\|$.

(3) $\nabla f(a,b)$ is perpendicular to the contour line of $f(x,y)$ through (a,b).

Why (3)? Since $f(x,y)$ is differentiable at (a,b), locally near (a,b) the graph of $z=f(x,y)$ is flat. Thus, locally contour lines are parallel:

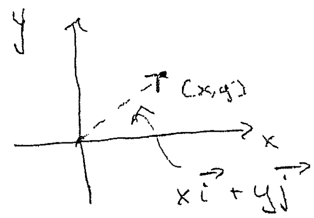


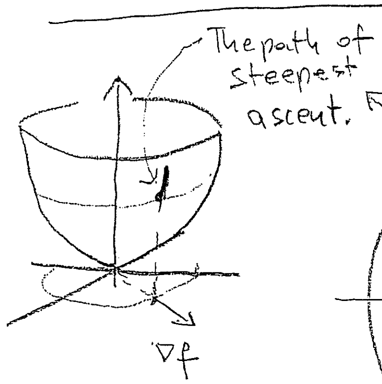
You obtain fastest growth by moving perpendicularly to contours in the direction of increasing values. So $\nabla f(a,b)$ that points toward fastest increase must be perpedicular to contours.

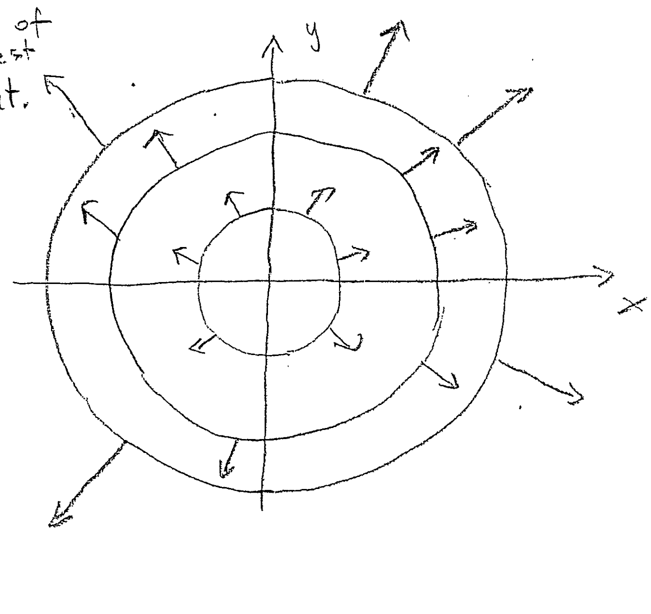
Ex: Let $f(x,y) = x^2 + y^2$. Sketch the contour diagram and the gradient field in one coordinate system.

$$\nabla f(x,y) = 2x\vec{i} + 2y\vec{j}$$

parallel the vector from $(0,0)$ to (x,y).







Magnitudes of vectors are scaled.

Ex A square metal plate is placed in the xy-plane in such a way that $0 \le x \le 3$, $0 \le y \le 3$ (All measurements in meters.). The temperature at each point (x,y) of the plane is given by:

$$T(x,y) = \frac{100}{x^2+y^2+1} \quad \text{in } {}^\circ F.$$

(a) Find the direction of the greatest increase in temp. at $(1,2)$. What is the greatest rate of increase at the point $(1,2)$?

(b) Find the rate of increase at $(1,2)$ in the direction $\vec{u} = \frac{2}{\sqrt{5}}\vec{i} + \frac{1}{\sqrt{5}}\vec{j}$.

Clearly, we have to find the gradient $\nabla T(x,y)$ first.

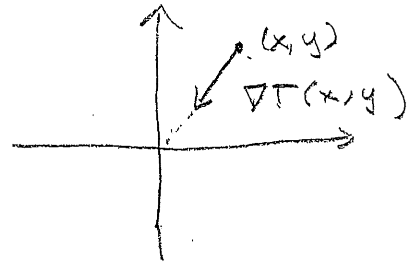
$$T_x(x,y) = \frac{\partial}{\partial x}\left[\frac{100}{x^2+y^2+1}\right] = \frac{\partial}{\partial x}\left[100(x^2+\underbrace{y^2+1}_{\text{Const}})^{-1}\right] =$$

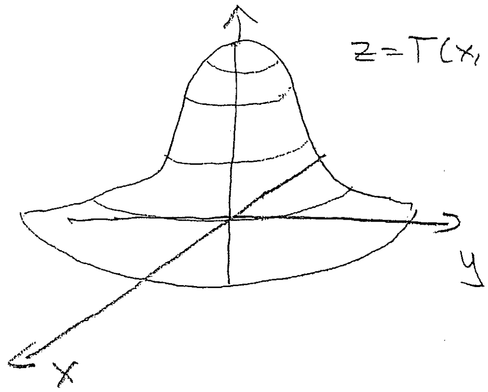
$$= 100 \cdot -(x^2+y^2+1)^{-2} \cdot 2x =$$

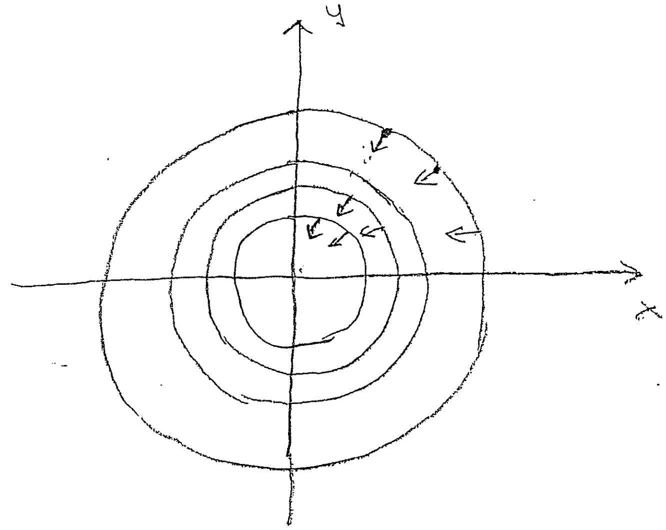
$$= -\frac{100}{(x^2+y^2+1)^2} \cdot 2x = T_x(x,y)$$

$$T_y(x,y) = -\frac{100}{(x^2+y^2+1)^2} \cdot 2y$$

$$\nabla T(x,y) = -\frac{100}{(x^2+y^2+1)^2}\left(2x\vec{i} + 2y\vec{j}\right)$$

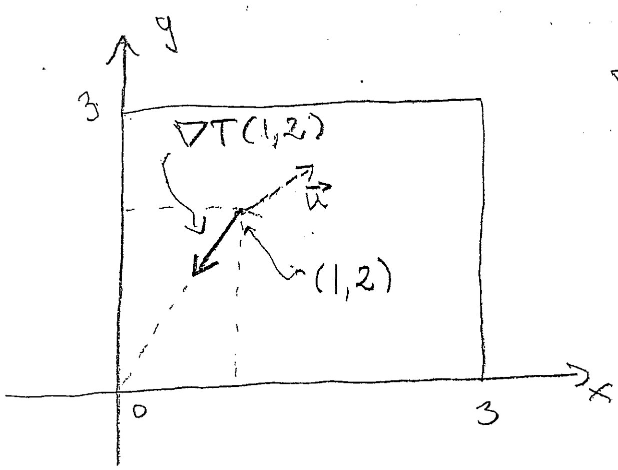






Contours are concentric circles centered at (0,0).

At each point the gradient $\nabla T(x,y)$ points toward the origin.



$$\nabla T(1,2) = -\frac{200}{36}\vec{i} - \frac{400}{36}\vec{j}$$

The direction of the greatest increase in temprature at (1,2). (Toward the origin). This answers (a).

(b) $\| \nabla T(1,2) \| = \sqrt{\left(\frac{200}{36}\right)^2 + \left(\frac{400}{36}\right)^2} \cong 77.16 \ \frac{°F}{m}$

(c) $T_{\vec{u}}(1,2) = \nabla T(1,2) \cdot \vec{u} =$

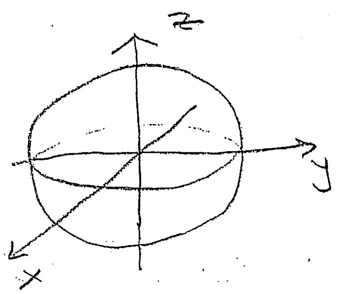
$$= -\frac{200}{36} \cdot \frac{2}{\sqrt{5}} - \frac{400}{36} \cdot \frac{1}{\sqrt{5}} \cong -9.93 \ \frac{°F}{m}$$

14.5 Gradients of Functions of Three Variables

Ex: $f(x,y,z) = x^2 + y^2 + z^2$

We cannot "graph" functions of three variables.
We can graph their level surface for a given k:

$$x^2 + y^2 + z^2 = k$$



We can, of course, define three partial derivatives at each point (a, b, c):

$$f_x(a,b,c), \quad f_y(a,b,c), \quad f_z(a,b,c)$$

by fixing two variables are differentiating with respect to the third. For example:

$$f_z(a,b,c) = \frac{d}{dz}\Big|_{z=c} [f(a,b,z)].$$

In our example:

$$f_x(x,y,z) = 2x, \quad f_y(x,y,z) = 2y, \quad f_z(x,y,z) = 2z.$$

We can define the directional derivative of a function of three variables $f(x,y,z)$ in the direction of $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$, $\|\vec{u}\| = 1$:

$$f_{\vec{u}}(a,b,c) = \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2, c+hu_3) - f(a,b,c)}{h}.$$

This is the rate of change in the direction of $\vec{u}$.

We can define the gradient vector:

$$\nabla f(x,y,z) = f_x(x,y,z)\vec{i} + f_y(x,y,z)\vec{j} + f_z(x,y,z)\vec{k}.$$

In our example, $f(x,y,z) = x^2+y^2+z^2$,

$$\nabla f = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}.$$

As before, if $f(x,y,z)$ is differentiable at (a,b,c):

$$f_{\vec{u}}(a,b,c) = \nabla f(a,b,c) \cdot \vec{u} \qquad \frac{\text{units of } f}{\text{units of dist in dir } \vec{u}}$$

A before:

If $\nabla f(a,b,c) \neq \vec{0}$, then

- $\nabla f(a,b,c)$ points in the direction of the greatest rate of change of f at (a,b,c).
- $\|\nabla f(a,b,c)\|$ is this greatest rate of change
- $\nabla f(a,b,c)$ is perpendicular to the level surface through (a,b,c).

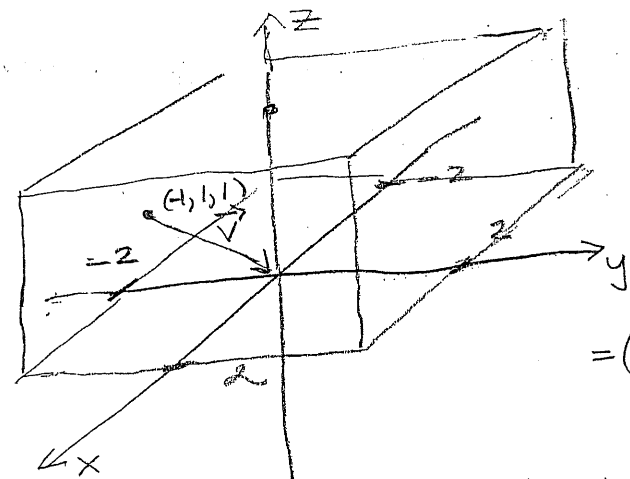
Ex Suppose that the function $F(x,y,z) = x^2 + y^4 + x^2 z^2$ gives concentration of salt, in gr/gal, at any point (x,y,z) of a rectangular tank of water occupying the region

$$-2 \le x \le -2, \quad -2 \le y \le 2, \quad 0 \le z \le 2,$$

(All measurments in meters.) Suppose you are at the point $(-1,1,1)$.

(a) In what direction should you move if you want the concentration to increase the fastest?

(b) If you move from $(-1,1,1)$ toward the origin $(0,0,0)$, how fast is the concentration changing?



(a)

$$\nabla F(x,y,z) = (2x + 2xz^2)\vec{i} + 4y^3\vec{j} + 2x^2 z\vec{k}$$

$$\nabla F(-1,1,1) = -4\vec{i} + 4\vec{j} + 2\vec{k}$$

The direction of greatest increase in concentration.

$$\|\nabla F(-1,1,1)\| = \sqrt{16+16+4} = 6 \ \frac{g/gal}{m}$$

(b) $\vec{v} = \overrightarrow{(-1,1,1)(0,0,0)} = \vec{i} - \vec{j} - \vec{k}, \quad \|\vec{v}\| = \sqrt{3}$

$$\vec{u} = \frac{1}{\sqrt{3}}\vec{i} - \frac{1}{\sqrt{3}}\vec{j} - \frac{1}{\sqrt{3}}\vec{k}$$

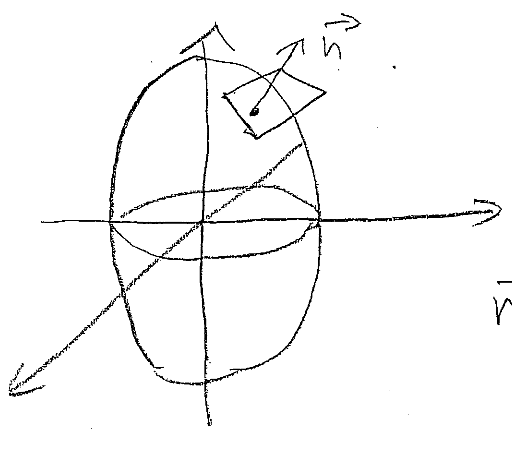
$$F_{\vec{v}}(-1,1,1) = F_{\vec{u}}(-1,1,1) = \nabla F(-1,1,1)\cdot\vec{u} =$$

$$= -\frac{4}{\sqrt{3}} - \frac{4}{\sqrt{3}} - \frac{2}{\sqrt{3}} = -\frac{10}{\sqrt{3}} \cong -5.77 \ \frac{g/gal}{m}.$$

Ex: Find the equation of the tangent plane to the ellipsoid $x^2+2y^2+z^2=15$ at $(2,1,3)$.

The ellipsoid is the level surface of

$$F(x,y,z) = x^2+2y^2+z^2$$



$$\vec{n} = \nabla F(2,1,3)$$

$$\nabla F = 2x\vec{i} + 4y\vec{j} + 2z\vec{k}$$

$$\vec{n} = \nabla F(2,1,3) = 4\vec{i} + 4\vec{j} + 6\vec{k}$$

$$P = (2,1,3)$$

Plane: $4(x-2)+4(y-1)+6(z-3)=0.$