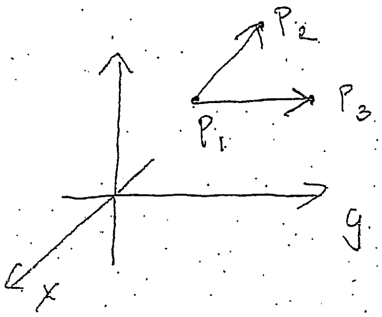
Ex: Find an equation of the plane passing through points

$P_1 = (1, 1, 1)$, $P_2 = (0, 1, 2)$, $P_3 = (2, -5, 0)$.

Find a unit normal vector to the plane.



We have a point, we need a normal vector. A normal will have to be $\perp$ to $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$.

$$\vec{n} = (\overrightarrow{P_1P_2}) \times (\overrightarrow{P_1P_3})$$

$$\overrightarrow{P_1P_2} = (-1, 0, 1), \quad \overrightarrow{P_1P_3} = (1, -6, -1)$$

$$\vec{n} = \begin{vmatrix} \vec{i}, & \vec{j}, & \vec{k} \\ -1 & 0 & 1 \\ 1 & -6 & -1 \end{vmatrix} = \vec{i} \cdot 6 - \vec{j} \cdot 0 + \vec{k} \cdot 6$$

$$\vec{n} = 6\vec{i} + 6\vec{k}$$

i) Equation (using P_1):

$$6(x-1) + 6(z-1) = 0$$

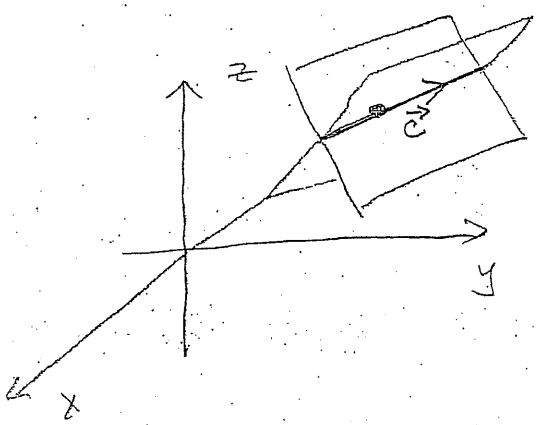
(b) $$\vec{u}_n = \frac{\vec{n}}{\|\vec{n}\|} = \frac{6}{6\sqrt{2}}\vec{i} + \frac{6}{6\sqrt{2}}\vec{k} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{k}$$

$$\|\vec{n}\| = \sqrt{36+36} = 6\sqrt{2}$$

Ex : Find a vector parallel to the intersection of the planes

L_1: $2x - 3y + 5z = 2$

L_2: $4x + y - 3z = 7$



$\vec{c} \perp n_{L_1}$, $\vec{c} \perp n_{L_2}$

$n_{L_1} = (2, -3, 5)$

$n_{L_2} = (4, 1, -3)$

$$\vec{c} = n_{L_1} \times n_{L_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 5 \\ 4 & 1 & -3 \end{vmatrix} =$$

$$= 4\vec{i} - (-26)\vec{j} + 14\vec{k} = \underline{4\vec{i} + 26\vec{j} + 14\vec{k}}.$$

14.1, 14.2 Partial Derivatives

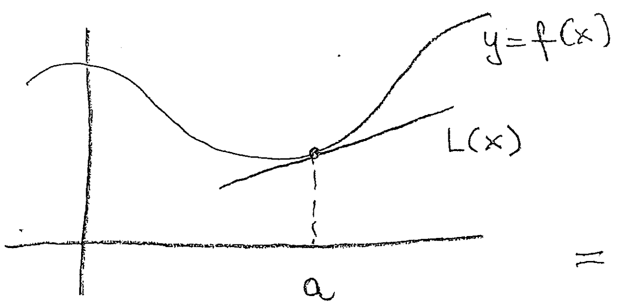
Going back to functions of two variables:

$$z = f(x,y).$$

Today we are going to define and interpret partial derivatives of $f(x,y)$ with respect to x and y.

Recall for a function of one variable:

$$y = f(x)$$



$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = m_{tan}$$

$$= \begin{pmatrix}\text{The rate of change} \\ \text{of } y \text{ at } x=a\end{pmatrix} \frac{\text{units of } y}{\text{unit of } x}$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

So $\quad f(x) \cong f(a) + f'(a)(x-a) \quad \leftarrow \quad (x = a+h)$

The equation of the tangent line:

$$L(x) = f(a) + f'(a)(x-a)$$

$L(x)$ local linearization of $f(x)$ at $x=a$.

$f(x)$ differentiable at $x=a \Leftrightarrow$ the tangent line exists.

How much of that translates to $f(x,y)$ and how?

The first step: partial derivatives.

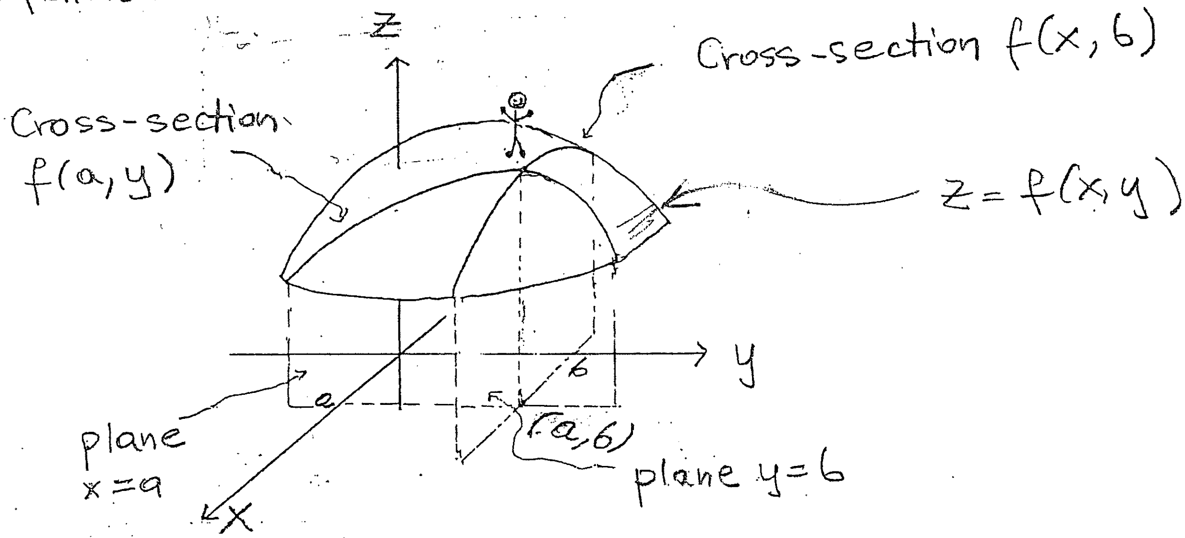
Let a function $z = f(x, y)$ and a point (a, b) in its domain be given. We define:

$f_x(a, b)$ – the partial derivative of $f(x, y)$ with respect to x at (a, b)

and

$f_y(a, b)$ – the partial derivative of $f(x, y)$ with respect to y at (a, b).

as follows.



$$f_x(a, b) = \frac{d}{dx}\Big|_{x=a} \left[f(x, b) \right]$$

$$f_y(a, b) = \frac{d}{dy}\Big|_{y=b} \left[f(a, y) \right]$$

In other words

$f_x(a, b)$ = the slope of $f(x, b)$ at $x = a$.

$f_y(a, b)$ = the slope of $f(a, y)$ at $y = b$.

So to calculate

$$f_x(a, b)$$

we fix $y = b$, consider the function $f(x, b)$ which is a function of one variable x, and calculate the ordinary derivative of $f(x, b)$ with respect to x.

Similarly, to calculate

$$f_y(a, b)$$

we fix $x = a$, consider $f(a, y)$ etc.

Ex: Let $f(x, y) = x^2 + y^3$. Find $f_x(2, 1)$, $f_x(1, 3)$.

To find $f_x(2, 1)$, we fix $y = 1$ and consider the cross-section

$$f(x, 1) = x^2 + 1$$

Now we take the ordinary derivative

$$\frac{d}{dx}[f(x, 1)] = \frac{d}{dx}[x^2 + 1] = 2x;$$

And evaluate at $x = 2$: $f_x(2, 1) = 2x\big|_{x=2} = 4$

For $f_x(1, 3)$, we take

$$f(x, 3) = x^2 + 27$$

$$\frac{d}{dx}\Big|_{x=1}[x^2 + 27] = 2x\big|_{x=1} = 2.$$

Clearly calculating $f_x(a,b)$, $f_y(a,b)$ for each given point separately is silly. For a given

$$f(x,y)$$

we should calculate partial derivatives functions:

$$f_x(x,y) \quad , \quad f_y(x,y)$$

and then evaluate at any point (a,b) we want. How to do this?

Ex: Let $f(x,y) = x^2 + y^3$. Find $f_x(x,y)$ and $f_y(x,y)$. Find $f_y(2,1)$.

To find $f_x(x,y)$, we take $f(x,y) = x^2 + y^3$ and assume that y is fixed so y is a constant. Assuming that y is a constant, we take the derivative of $f(x,y) = x^2 + y^3$ with respect to x:

$$f_x(x,y) = (x^2 + y^3)_x = 2x$$

To find $f_y(x,y)$, we assume that x is a constant and differentiate with respect to y:

$$f_y(x,y) = (x^2 + y^3)_y = 3y^2.$$

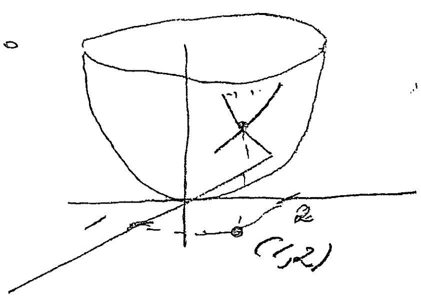
So $f_y(2,1) = 3$. Easy!

Ex: Let $f(x,y) = x^2+y^2$. Find $f_x(1,2)$, $f_y(1,2)$.

$f_x(x,y) = 2x$, $f_y(x,y) = 2y$

$f_x(1,2) = 2$, $f_y(1,2) = 4$.

So



For the paraboloid at (1,2) the slope in the x direction is 2, the slope in the y direction is 4.

Leibnitz Notation:

$z = f(x,y)$, $f_x(x,y) = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[f(x,y)]$,

$f_y(x,y) = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[f(x,y)]$

$f_x(a,b) = \frac{\partial z}{\partial x}\Big|_{(a,b)}$, $f_y(a,b) = \frac{\partial z}{\partial y}\Big|_{(a,b)}$

Ex: $z = x^2 y$. Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[x^2 y] = 2yx$, $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[x^2 y] = x^2$.

For a given $z = f(x, y)$, find partial derivatives $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

Ex: $z = \sin(x y^3)$.

$$\frac{\partial z}{\partial x} = \cos(x y^3) \cdot y^3 \qquad \left(\sin(x \cdot 5)\right)'_x$$

$$\frac{\partial z}{\partial y} = \cos(x y^3) \cdot 3 x y^2 \qquad \left(\sin(5 y^3)\right)'_x$$

Ex: $z = x^2 e^{x \cdot y}$

$$\frac{\partial z}{\partial x} = 2x\, e^{xy} + x^2 y e^{xy}$$

$$\frac{\partial z}{\partial y} = x^2 \cdot x\, e^{xy} = x^3 e^{xy}$$

Practice!

Of course partial derivatives, $f_x(a, b)$ $f_y(a, b)$ are rates of change of $f(x, y)$ at (a, b) in the x direction any the y direction.